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GENERAL THEORY AND METHODS FOR MATRIC FACTORING

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Methods are developed for factoring an arbitrary rectangular matrix S of rank r into the form FP, where F has r columns and P has r rows. For the statistical problem of factor analysis, S may be the score matrix of a population of individuals on a battery of tests. Then F is a matrix of factor loadings, P is a matrix of factor scores, and r is the number of factor variates. (As in current procedures, there remains a subsequent problem of rotation of axes and interpretation of factors, which is not discussed here.) Methods are also developed for factoring an arbitrary Gramian matrix G of rank r into the form FF', where F has r columns and F' denotes F transposed. For the statistical problem of factor analysis, G may be the matrix of intercorrelations, R, of a battery of tests, with unity, communalities, or other parameters in the principal diagonal. R is proportional to SS', and it is shown that S can be factored by factoring R. This may usually be the most economical procedure in practice; it should not be overlooked, however, that S can be factored directly. The general methods build up an F (and P) in as many stages as desired; as many factors as may be deemed computationally practical can be extracted at a time. Perhaps it will usually be found convenient to extract not more than three factors at a time. Current procedures, like the centroid and principal axes, are special cases of a general method presented here for extracting one factor at a time.

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1. *Introduction*

This paper develops a variety of new methods for extracting factors from matrices. It gives methods for extracting one factor at a time and methods for extracting several factors at a time, be they oblique or orthogonal. Prevalent methods, like the centroid and principal axes, are shown to be special cases of those developed here.

As an example of methods new to factor analysis, it is shown that "weighted centroid" methods can be used instead of the centroid method, that there is no need to be restricted to ± 1 weights in summing columns in the correlation matrix to determine factor loadings. By judicious choice of weights, according to their particular data, research workers can reduce mean square residuals more rapidly at each stage than by present routines, and without a disproportionate increase in labor.

A variety of new techniques is given for extracting *more* than one factor at a time. It should be feasible to routinize the extraction of two or three factors at a time by the formulas developed below.

Many other possible variations in technique follow from the general formulas established here.

The methods presented are actually adaptations and generalizations of a technique due to Lagrange for reducing bilinear and quadratic forms (**10**, p. 68). They are justified by three theorems established below (see §§3, 7, and 8), one of which is a generalization of that of Lagrange. These three theorems provide a complete theory of matric factoring for factor analysis.

The formulas are derived rigorously. Experience has shown that lack of attention to rigor often leads to unsuspected pitfalls in statistical methods for the social and psychological sciences. Factor analysis in particular has now reached a stage which makes it necessary that the discipline be put on a more rigorous foundation. As an example, in the Spearman-Thurstone approach there are unsolved problems associated with the determination of communalities which—when solved—may conceivably vitiate a considerable amount of published empirical work. In this paper, we shall assume that the problems of communalities can be taken care of, so that the factoring of matrices in the Spearman-Thurstone approach is included.

Attention to rigor has other practical benefits. To my knowledge, no complete justification has been published before for the factoring method most widely in use—Thurstone's centroid method (**9**). It was trying to prove this and related things that led to the uncovering of the more general methods presented here.

Thurstone's centroid method is partly justified—assuming the communality problems are cleared up if communalities are used—as a special case of Lagrange's method. As is shown below, Lagrange's theorem proves that the centroid method does reduce the rank of a Gramian matrix by unity at each stage. Completion of the justification is provided by theorems (2) and (3) of §7 and §8 below.

Because of the practical and theoretical suggestions forthcoming from such a treatment, the theory of factoring is presented first from

a general matric point of view. The case of the factoring of an arbitrary rectangular matrix is discussed first. This then includes the direct factoring of a test score matrix. The methods are seen to apply in particular to Gramian matrices, of which correlation matrices are a special case.

2. *Some Properties of Gramian Matrices*

In the demonstrations to be given below, certain properties of Gramian matrices will be utilized. Proof of these properties can be found elsewhere in a systematic treatment of Gramian matrices (3), so we shall simply list here those theorems assumed in the present paper. The theorems will be stated here only as for real matrices for convenience.* The transpose of a matrix $\boldsymbol{M}$ is denoted by $\boldsymbol{M}'$.

Definition.† A Gramian matrix is a symmetric matrix in which all principal minors of all orders are nonnegative. (3, p. 49)

Theorem (*a*). If $\boldsymbol{M}$ is any rectangular matrix of rank r, then $\boldsymbol{MM}'$ is Gramian and of rank r. (3, p. 53)

Theorem (*b*). For any Gramian matrix $\boldsymbol{G}$ of order $n \times n$ and of rank $r > 0$ there exists an $\boldsymbol{F}$ of order $n \times r$ (and of rank r) such that $\boldsymbol{FF}' = \boldsymbol{G}$. (3, p. 55)

Theorem (*c*). If $\boldsymbol{G}$ is Gramian and nonsingular, then all principal minors of $\boldsymbol{G}$ of all orders are positive. (3, p. 56)

Theorem (*d*). If $\boldsymbol{M}$ is any matrix whose number of columns equals the order of $\boldsymbol{G}$, where $\boldsymbol{G}$ is Gramian, then $\boldsymbol{MGM}'$ is Gramian (3, p. 75).

Theorem (*e*). Every Gramian matrix has a unique Gramian square root. (3, p. 121).

Theorem (*f*). If $\boldsymbol{MM}' = \boldsymbol{FF}' = \boldsymbol{G}$, where $\boldsymbol{F}$ is as defined in theorem (b), then there exists an orthogonal matrix $\boldsymbol{U}$ such that

$$\boldsymbol{M} = \| \boldsymbol{F}\, 0 \| \boldsymbol{U} \qquad (3, \text{p. } 68).$$

3. *The Generalization of Lagrange's Theorem*

In this and the next section, the problem of factoring will be dealt with from a purely matric point of view, and special treatment will be made later of the case of statistical factors.

We first need to establish the following lemma.

Lemma 1. If S *is of order* n × N *and of rank* r > 0, *then for any in-*

* The treatment in (3) is for general complex Gramian matrices; and actually, the theorems for matric factoring of the present paper hold with but a change in wording for complex matrices.

† An equivalent definition, as can be seen from theorems (a) and (b), is as follows: a matrix expressible in the form $\boldsymbol{MM}'$. Other equivalent definitions are possible.

teger s *such that* $0 < s \leqq r$ *there exists an infinite number of pairs of matrices* **X** *and* **Y** *of orders* s×n *and* s×N, *respectively, such that* **XSY′** *is nonsingular. In fact, all nonsingular matrices of order* s×s *are expressible in the form* **XSY′**.

The proof follows from a theorem in linear dependence which states (1, p. 67) that there always exist nonsingular matrices A_n and A_N, of orders $n \times n$ and $N \times N$, respectively, such that

$$A_n S A'_N = \begin{Vmatrix} I_r & 0 \\ 0 & 0 \end{Vmatrix},$$

I_r being the unit matrix of order $r \times r$. Let X_0 be the first s rows of an A_n, and let Y_0 the first s rows of a corresponding A_N. Then $X_0 S Y'_0 = I_s$, where I_s is the unit matrix of order $s \times s$. Let B and C be arbitrary nonsingular matrices of order $s \times s$, and let

$$X = BX_0, \qquad Y = CY_0.$$

Then

$$XSY' = BX_0 S Y'_0 C' = BI_s C' = BC'.$$

Clearly the last member is nonsingular, so that the first member is nonsingular.

Since A_n and A_N can be determined in infinitely many ways, so can X_0 and Y_0, and all the more so can X and Y. Moreover, since B and C are arbitrary, any nonsingular* matrix of order $s \times s$ can be expressed in the form BC', hence in the form XSY'.

The following theorem is basic to the entire discussion.

Theorem 1. Let S *be any matrix of order* n×N *and of rank* $r > 0$. *Let* **X** *and* **Y** *be of orders* s×n *and* s×N, *respectively (where* $s \leqq r$), *and such that* **XSY′** *is nonsingular. Then the residual matrix*

$$S_1 = S - SY'(XSY')^{-1}XS \qquad (1)$$

is exactly of rank $r - s$.

This theorem generalizes a result due to Lagrange (10, p. 68) which can be stated as follows:

Lagrange's Theorem: Let S *be any matrix of order* $n \times n$ *and of rank* $r > 0$. *Let* x *and* y *be row vectors of* n *elements each and such that* $xSy' \neq 0$. *Then the residual matrix*

$$S_1 = S - \frac{Sy'\,xS}{xSy'}$$

is exactly of rank $r - 1$.

* More generally, if we do not restrict B and C to being nonsingular, we prove that *any* matrix of order $s \times s$ is expressible in the form XSY'.

For the proof of theorem (1) (which will prove Lagrange's theorem in particular), we first note that lemma (1) assures the existence of an infinite number of satisfactory matrices X and Y. Form the supermatrix

$$M = \left\| \begin{matrix} I_s & (XSY')^{-1}XS \\ SY' & S \end{matrix} \right\|,$$

where I_s is again the unit matrix of order $s \times s$, and M is of order $(n+s) \times (N+s)$. Postmultiplying the first submatric column of M by $(XSY')^{-1}XS$ and subtracting from the second column, and using (1), yields

$$M_1 = \left\| \begin{matrix} I_s & 0 \\ SY' & S_1 \end{matrix} \right\|.$$

Premultiplying the first row of M_1 by SY' and subtracting from the second row yields

$$M_2 = \left\| \begin{matrix} I_s & 0 \\ 0 & S_1 \end{matrix} \right\|.$$

Premultiplying the second submatric row of M by $(XSY')^{-1}X$ and subtracting from the first row yields

$$M_3 = \left\| \begin{matrix} 0 & 0 \\ SY' & S \end{matrix} \right\|.$$

Postmultiplying the second column of M_3 by Y' and subtracting from the first column yields

$$M_4 = \left\| \begin{matrix} 0 & 0 \\ 0 & S \end{matrix} \right\|.$$

Now M, M_1, M_2, M_3, and M_4 differ from each other only by elementary transformations, so that they all have the same rank. By inspection, the rank of M_2 is the rank of S_1 plus s (the rank of I_s); and the rank of M_4 is r, the rank of S. Therefore, the rank of S_1 is exactly $r - s$.

An important additional fact is that

$$XS_1 = 0, \qquad S_1Y' = 0, \tag{2}$$

as is readily verifiable by premultiplying both members of (1) by X, and by postmultiplying each member by Y'. Therefore, if it is desired to "factor" S_1 further by the method of the theorem just established,

neither X nor Y can be used. The condition that $X_1S_1Y'_1$ be nonsingular, where the matrices with subscripts now are to replace those without subscripts in theorem (1), would not be satisfied if X and/or Y is used for X_1 and/or Y_1.

By choosing X_1 and Y_1 of the appropriate rank and such that $X_1S_1Y'_1$ is nonsingular, the rank of S_1 can be reduced again by any desired amount, leaving a residual matrix S_2. The process can be continued until a residual matrix of zero is obtained.

4. *The Factor Matrices*

A matrix S, of order $n \times N$ and of rank $r > 0$, will be said to be factored by matrices F and P if F and P are of orders $n \times r$ and $r \times N$, respectively, and such that $S = FP$. Then F and P are necessarily of rank r, for their ranks cannot exceed r because of their orders, and the rank of S cannot be greater than either of their ranks since S is their product.

Evidently, if one pair F and P exists, then there is an infinite number of pairs of such factor matrices for a given S. If Z is an arbitrary nonsingular matrix of order $r \times r$, then

$$F_0 = FZ^{-1}, \qquad P_0 = ZP$$

will define a new pair of factor matrices, for F_0 and P_0 are of the requisite rank and orders, and $S = F_0P_0$.

By using theorem (1), we can build up an F and P in stages, incidentally proving that *every* S *can be factored.* In theorem (1), let

$$F_1 = SY'(XSY')^{-1}Z_1^{-1}, \tag{3}$$

where Z_1 is an arbitrary nonsingular matrix of order $s \times s$, and let

$$P_1 = Z_1XS. \tag{4}$$

Then equation (1) can be rewritten as

$$S_1 = S - F_1P_1. \tag{5}$$

Subsequent residual matrices can be similarly treated until, if the mth residual matrix is the first to be zero, we have

$$\begin{aligned} S_2 &= S_1 - F_2P_2 \\ S_3 &= S_2 - F_3P_3 \\ &\cdots\cdots \\ 0 &= S_{m-1} - F_mP_m. \end{aligned}$$

Adding corresponding members of these m equations, we see that

$$S = F_1P_1 + F_2P_2 + \cdots + F_mP_m.$$

Therefore, if we form the supermatrices

$$\boldsymbol{F} = ||\boldsymbol{F}_1\boldsymbol{F}_2 \cdots \boldsymbol{F}_m||, \qquad \boldsymbol{P} = \left\| \begin{array}{c} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \cdots \\ \boldsymbol{P}_m \end{array} \right\|,$$

then $\boldsymbol{F}$ and $\boldsymbol{P}$ will be factor matrices of $\boldsymbol{S}$, since they are of order $n \times r$ and $r \times N$, respectively, by construction, and $\boldsymbol{S} = \boldsymbol{FP}$.

5. *The Problem of Metric*

In the statistical problem of factor analysis, the matrix $\boldsymbol{S}$ to be factored is a matrix of scores of N individuals on n tests. These scores are usually expressed in standard form: with zero means and unit variances. Or else we can consider the case where unique factor scores are subtracted out, so that $\boldsymbol{S}$ represents reduced test scores, with zero means again, but with communalities as variances.

A factor matrix $\boldsymbol{P}$ represents the scores of the N individuals on a set of r (common) factors, and these factor scores are usually desired to be expressed in standard form. The corresponding factor matrix $\boldsymbol{F}$ is then said to contain the loadings of the n tests on the r factors.

We can easily show that if the means of the test scores are all zero, then the means of the factor scores in $\boldsymbol{P}_1$ are all zero. Form the row vector of N elements, each of which is unity,

$$\Sigma_N = ||1\,1\,1 \cdots 1||.$$

To say that the test scores have zero means is to say that

$$\boldsymbol{S}\Sigma'_N = 0\,. \tag{6}$$

Postmultiplying both members of (4) by Σ'_N, and using (6),

$$\boldsymbol{P}_1\Sigma'_N = 0\,, \tag{7}$$

or the s sets of factor scores have zero means.

Postmultiplying both members of (5) by Σ'_N, and using (6) and (7), we see that the right member becomes zero, so that the means of the rows in $\boldsymbol{S}_1$ are all zero. Hence, the means of the factor scores in $\boldsymbol{P}_2$ are zero. Similarly, we see that the means of the rows of all the remaining residual matrices are zero, and the means of all the remaining factor scores are zero.

It is quite another matter, however, to discuss the variances of the factor scores, and even more so the intercorrelations between factors. If the tests are in standard form, then their correlation matrix is

$$\boldsymbol{R}=\frac{1}{N}\boldsymbol{SS}', \tag{8}$$

the diagonal elements being unity. If the tests have unique factor scores subtracted out, then the diagonal elements in (8) are communalities. The covariance matrix of the s factor scores in $\boldsymbol{P}_1$ (which we now know to have zero means) is, according to (4), and using (8),

$$\frac{1}{N}\boldsymbol{P}_1\boldsymbol{P}'_1=\frac{1}{N}\boldsymbol{Z}_1\boldsymbol{XSS}'\boldsymbol{X}'\boldsymbol{Z}'_1=\boldsymbol{Z}_1\boldsymbol{XRX}'\boldsymbol{Z}'_1. \tag{9}$$

In order for these factor scores to have unit variances, the principal diagonal elements of the covariance matrix must all be unity. One way of achieving this—as can be seen from the last member of (9)—would be to let $\boldsymbol{Z}_1$ be a diagonal matrix whose diagonal elements are reciprocals* of the (positive) square roots of the corresponding principal diagonal elements of $\boldsymbol{XRX}'$. Another way would be to modify any given $\boldsymbol{Z}_1$ by premultiplying it by a diagonal matrix whose elements are reciprocals* of the (positive) square roots of the corresponding elements in the principal diagonal of the last member of (9), and using this modified matrix to replace the old $\boldsymbol{Z}_1$.

In order to obtain *orthogonal* or *uncorrelated* factor scores, the left member of (9) would have to be the unit matrix. Therefore, in order to obtain orthogonal factors, $\boldsymbol{Z}_1$ would have to be determined to make the last member of (9) the unit matrix.

A similar discussion holds separately for each of $\boldsymbol{P}_2, \boldsymbol{P}_3, \cdots, \boldsymbol{P}_m$.

There doesn't seem to be much profit in discussing here methods for getting orthogonal factors for the most general case, so this topic is not pursued further here. Instead, we go on to the following special method which is of considerable importance.

6. *Orthogonal Factors*

The following lemma will be needed here.

Lemma 2. If $\mathbf{G}$ *is Gramian, of order* $\mathrm{n}\times\mathrm{n}$, *and of rank* $\mathrm{r}>0$, *then for any integer* $\mathbf{s}$ *such that* $0 < \mathrm{s} \leqq \mathrm{r}$ *there exists an infinite number of matrices* $\mathbf{X}$ *of order* $\mathrm{s}\times\mathrm{n}$ *such that* $\mathbf{XGX}'$ *is nonsingular. In fact,*

* The diagonal elements are not zero and are actually positive. To see this, note that the rank of $\boldsymbol{P}_1\boldsymbol{P}'_1$ must be the rank of $\boldsymbol{P}_1$ [theorem (a)], so that the covariance matrix is a nonsingular Gramian matrix. Hence all principal minors must be positive [theorem (c)]; in particular, the principal diagonal elements must be positive. $\boldsymbol{XRX}'$ is also Gramian since $\boldsymbol{R}$ is Gramian [theorem (d)], and is nonsingular since it differs from $\boldsymbol{P}_1\boldsymbol{P}'_1$ only by nonsingular transformations. Hence its diagonal elements are also all positive.

all nonsingular Gramian matrices of order s×s *are expressible in the form* **XGX′**.

The proof follows from the fact that any symmetric matrix can be diagonalized by an orthogonal matrix (10, p. 90). Thus, there exists an orthogonal matrix U such that

$$UGU' = \begin{Vmatrix} D^2_r & 0 \\ 0 & 0 \end{Vmatrix},$$

where D_r is a diagonal matrix whose diagonal elements are the (positive) square roots of the r positive latent roots of G. [That the nonzero latent roots of G must all be positive is well known. To verify this directly, note that D^2_r must be Gramian, since UGU' must be Gramian by theorem (d).] Form the $n \times n$ matrix

$$A = \begin{Vmatrix} U_r D_r^{-1} & A_1 \\ 0 & A_2 \end{Vmatrix},$$

where U_r is an arbitrary orthogonal matrix of order $r \times r$, A_1 is an arbitrary matrix of order $r \times (n - r)$, and A_2 is an arbitrary matrix of order $(n - r) \times (n - r)$. It can be verified by direct multiplication that

$$A(UGU')A' = \begin{Vmatrix} I_r & 0 \\ 0 & 0 \end{Vmatrix}.$$

Let X_0 be the first s rows of AU. Then $X_0 G X'_0 = I_s$. Let B be an arbitrary nonsingular matrix of order $s \times s$, and let $X = BX_0$. Then

$$XGX' = BX_0 GX'_0 B' = BI_s B' = BB'.$$

Since the last member is nonsingular, the first must be.

Because of the arbitrary submatrices in A, X_0 can be determined in an infinite number of ways (assuming $n > 1$ if $r = n$), and hence all the more so can X. Since B is arbitrary, any nonsingular* Gramian matrix of order $s \times s$ can be expressed in the form BB' by theorem (b), and hence can be expressed in the form XGX'.

Returning to theorem (1), instead of considering an arbitrary Y, let

$$Y = XS, \tag{10}$$

* More generally, if we do not restrict B to being nonsingular, we prove that *any* Gramian matrix of order $s \times s$ is expressible in the form XGX'.

where X is still arbitrary, except again for the restriction that

$$XSY' = XSS'X' \tag{11}$$

be nonsingular. SS' is Gramian and of rank r by theorem (a), so that the existence of many such matrices X is assured by lemma (2). Using (10) and (11) in (3) and (4), and letting*

$$Z_1 = \sqrt{N}(XSS'X')^{-\frac{1}{2}},$$

we get

$$F_1 = \frac{1}{\sqrt{N}} SS'X'(XSS'X')^{-\frac{1}{2}} \tag{12}$$

$$P_1 = \sqrt{N}(XSS'X')^{-\frac{1}{2}}XS. \tag{13}$$

Then

$$\frac{1}{N} P_1P'_1 = (XSS'X')^{-\frac{1}{2}}XSS'X'(XSS'X')^{-\frac{1}{2}} = I.$$

The last member shows that the factor scores have unit variances and are orthogonal. Therefore (12) and (13) provide formulas for direct computations of orthogonal factors.

Note that, using (8), we can rewrite (12) and (13) as

$$F_1 = RX'(XRX')^{-\frac{1}{2}} \tag{14}$$

$$P_1 = (XRX')^{-\frac{1}{2}}XS. \tag{15}$$

These are more compact formulas for direct computations of orthogonal factors.

Some additional facts of some interest, although not very relevant here, are as follows. Postmultiplying both members of (15) by $\frac{1}{N}S'$, and again using (8), we see that

$$\frac{1}{N} P_1S' = (XRX')^{-\frac{1}{2}}XR,$$

or, comparing the right member with that of (14),

$$\frac{1}{N} P_1S' = F'_1. \tag{16}$$

Equation (16) states that the factor loadings are the *covariances* between tests and factors.

* The Gramian square root of $XSS'X'$ could be used, for example [theorem (e)].

An additional bit of information can be got by premultiplying (14) by X, which yields

$$XF_1 = (XRX')^{-1} = F'_1X' . \tag{17}$$

Formulas similar to (12) and (13) hold for $F_2, P_2, \cdots, F_m, P_m$, yielding factors which are orthogonal *within* each set. However, with no further restrictions on the X_j, the factors will not in general be orthogonal *between* sets. That is, in general, if $j \neq k$,

$$P_jP'_k \neq 0 ,$$

whereas if $j = k$,

$$\frac{1}{N} P_jP'_j = I.$$

In the computational methods based on the correlation matrix, to be described next, factors will be orthogonal between as well as within sets.

7. *The Justification for Computations Based on the Correlation Matrix*

Current techniques for factoring operate on the correlation matrix R rather than on the score matrix S. The usual purpose is to find an F of order $n \times r$ and of rank r such that $R = FF'$.

As we shall see, this implies orthogonal factor scores. In order to justify such a procedure, it is necessary to prove the following theorem:

Theorem 2. If

$$R = \frac{1}{N} SS' = FF' , \tag{18}$$

where S *is of order* n × N *and of rank* r, *and* F *is of order* n × r (*and of rank* r), *then it is possible to determine a* P *of order* r × N *and such that*

$$S = FP. \tag{19}$$

Furthermore, this P *is uniquely determined, and it satisfies*

$$\frac{1}{N} PP' = I. \tag{20}$$

The existence of a P satisfying (19) follows from theorem (f), setting $\frac{1}{\sqrt{N}} S = M$, and letting $\frac{1}{\sqrt{N}} P$ be the first r rows of U. Once

the existence of such a P is established,* the fact that it is unique can be immediately shown by premultiplying both members of (19) by† $(F'F)^{-1}F'$, yielding

$$P = (F'F)^{-1}F'S . \tag{21}$$

The right member of (21) is clearly unique for a given F.

That P satisfies (20) follows from the fact that U is orthogonal. Actually, we can postmultiply each member of (21) by its transpose, divide by N, and use (18):

$$\frac{1}{N}PP' = \frac{1}{N}(F'F)^{-1}F'SS'F(F'F)^{-1} = (F'F)^{-1}F'FF'F(F'F)^{-1} = I.$$

This is direct verification that P satisfies (20).

This last result proves that no matter how the F is built up, the corresponding factor scores are automatically orthogonal as long as $R = FF'$. Hence, if such an F is built up by subtests, as is done in the following remaining pages, the corresponding factor scores are automatically orthogonal both within and between subsets.

8. *Factoring the Correlation Matrix*

Current techniques, like Thurstone's centroid method, usually build up a matrix F by computing one column at a time. In the terminology of factor analysis, one factor is extracted at a time.

From theorem (1), we can get a method for extracting *as many factors as we wish at a time*, operating directly on the correlation matrix. Since this will be a general method for factoring any Gramian matrix, let us express it in general terms.

The fundamental theorem for the method is as follows:

Theorem 3. Let G *be a Gramian matrix of order* n×n *and of rank* $r > 0$. *Let* X *be of order* s×n *and such that* XGX′ *is nonsingular. Then the residual matrix*

$$G_1 = G - GX'(XGX')^{-1}XG \tag{22}$$

is of rank r − s *and is Gramian.*

The proof of the first half of the theorem, that the rank of G_1 is $r - s$, is given by theorem (1), substituting G for S and letting $Y = X$. Lemma (2) assures the existence of infinitely many satisfactory matrices X.

* Holzinger and Harman (4, pp. 289-290) tackle a problem similar to that of theorem (2) in attempting to show the relations between sets of factors. However, what they actually accomplish is to provide a formula, resembling (21) above, for transforming one solution into another *if the transformation is possible.* But the problem is to show that the transformation is possible. Proof of this follows from theorem (f).

† $F'F$ is nonsingular since its rank must be that of F [theorem (a)].

To prove the second half, that G_1 is Gramian, let $G = F_0F'_0$, where F_0 is of order $n \times r$, according to theorem (b). Then (22) can be rewritten as

$$\begin{aligned} G_1 &= F_0F'_0 - F_0F'_0X'(XF_0F'_0X')^{-1}XF_0F'_0 \\ &= F_0[I - F'_0X'(XF_0F'_0X')^{-1}XF_0]F'_0. \end{aligned}$$

In the last member, denote the symmetric matrix in brackets by H. It is readily verified by direct multiplication that $H = H^2$, or we can write

$$H = HH'. \tag{23}$$

From theorem (a), the right member of (23) is Gramian, so that H is Gramian. Since $G_1 = F_0HF'_0$, G_1 must also be Gramian from theorem (d).

To obtain s columns of a factor matrix F for G, we can let

$$F_1 = GX'(XGX')^{-1},$$

which is of order $n \times s$, and where the Gramian square root, say, of XGX' is used. Then (22) can be rewritten as

$$G_1 = G - F_1F'_1.$$

Since G_1 is Gramian, the factoring process can be reapplied to G_1, and similarly to subsequent residual matrices, until the remaining $r - s$ columns of an F are established.

As a special case of equation (2), we note that

$$XG_1 = 0, \tag{24}$$

so that new X_j must be used each time for each subset.

For the case of statistical factors, the Gramian matrix G is the correlation matrix R, with usually either unity or communalities in in the diagonal.

9. *The Special Case of the Centroid Method*

The calculation of $(XGX')^{-1}$ may in general constitute the major obstacle to successful application of the preceding method when $s > 3$. It should not be difficult to work out relatively simple procedures for the case $s \leqq 3$.

In the special case where $s = 1$, the work is extremely simple. X becomes simply a row vector, say x, and xGx' is simply a positive scalar number. Hence $(xGx')^{-1}$ is easily computed by ordinary arithmetic. In fact, we can write F_1 as a column vector, say

$$f_1 = \frac{Gx'}{\sqrt{xGx'}}. \tag{25}$$

In the special case where the elements of x are chosen so as to be all unity, say

$$x = \Sigma_n = ||1\ 1 \cdots 1||,$$

we have *Thurstone's centroid method,* where

$$f_1 = \frac{G\Sigma'_n}{\sqrt{\Sigma_n G \Sigma'_n}}.$$

As a special case of (24), we have $\Sigma_n G_1 = 0$, so that Σ_n cannot be used for subsequent factoring, as is well known in Thurstone's method.

In Thurstone's method, reflections of axes are made for the extraction of subsequent factors. This simply means that for factoring the residual matrices, vectors of the type

$$x_j = ||\pm 1\ \pm 1\ \pm 1 \cdots \pm 1||$$

are used, where the signs may be chosen in any manner, just so $x_j G_j x'_j \neq 0$ (in which case $x_j G_j x'_j > 0$, since G_j is nonnegative definite).

The present analysis shows that *there is no need to be restricted to* x_j *whose elements are* ± 1. Especially if Thurstone's centroid is used for the first factor loadings, it seems that the elements of subsequent residual matrices will in general tend to diminish more rapidly in the sense of least squares by judicious choice of x_j with an eye to maximizing $x_j G_j x'_j / x_j x_j'$ (cf. 8).

For the statistical problem of factor analysis, further investigation of this method should yield dividends in practical procedures that will reduce mean square residuals rapidly at each stage, without involving the labor of computing principal axes. An alternative goal would be to try to arrive as closely as possible to a simple structure —if it exists— in the process of extracting the factors, so as to minimize a subsequent problem of rotation.

10. *The Special Case of Principal Axes*

As is well known, the method of principal axes minimizes the mean square residual at each stage (6, 8). We shall develop this method here as a special case of Lagrange's method.

Returning to the matrix S and §6, consider the case where $s = 1$, so that X and Y are row vectors, say x and y. Suppose that x and y are chosen to satisfy the simultaneous matric equations

$$xS = \lambda y \tag{26}$$

$$yS' = \mu x, \tag{27}$$

where λ and μ are scalars to be determined. Substituting $1/\lambda$ times the left member of (26) for y in (27), and multiplying through by λ,

$$xSS' = \lambda\mu x . \tag{28}$$

This shows that x is a latent vector of SS' (and hence of R), and $\lambda\mu$ is a latent root of SS'.

Similarly, by substituting $1/\mu$ times the left member of (27) for x in (26), and multiplying through by μ,

$$yS'S = \lambda\mu y . \tag{29}$$

Hence* y is a latent vector of $S'S$ corresponding to the same latent root $\lambda\mu$ as does x.

Equations (26) and (27) can be used as a basis for an iterative method for principal axes that operates on the *test scores* rather than on the correlation matrix. This would be of advantage, for instance, when the intercorrelations between the tests are not desired, but only the major principal axis is sought. In such a case, it may be less work to perform iterations directly on the scores rather than to compute all the intercorrelations first in order to perform iterations on R.

Letting x_k, y_k, λ_k, and μ_k be values of the trial vectors and constants of proportionality at the kth trial, the iterations take the form

$$\lambda_k y_k = x_k S , \quad \mu_{k+1} x_{k+1} = y_k S' .$$

Proof that convergence will take place in general to the major axis follows immediately from the proof for iterations on the correlation matrix (cf. 6, 8), as inspection of the transition from equations (26) and (27) to (28) and (29) will show.

When $s > 1$, iterations on the score matrix may not be too feasible, for the general case, to get several principal axes simultaneously.

Returning to the factoring of a Gramian matrix—like the correlation matrix— when $s = 1$, we see that a principal axis is a special case of (25), where now x satisfies an equation

$$xG = \xi^2 x ,$$

ξ^2 being a latent root of G. If x is normalized,

$$xGx' = \xi^2 xx' = \xi^2 ,$$

* We have tacitly assumed that $\lambda\mu \neq 0$. (28) and (29) obviously are true if $\lambda\mu = 0$. There will be $n-r$ independent solutions to (28) for $\lambda\mu = 0$, and there will be $N-r$ for (29). All such solutions, however, are of no interest for the present factor problem, since only those r solutions corresponding to the r positive latent roots go into the factor matrices.

so that (25) can be written in this case as

$$f_1 = \xi x' .$$

Hotelling has discussed iterations on the correlation matrix for the case $s = 1$ (6, 7).* Duncan, Frazer, and Collar (2) have discussed this type of iteration for a more general matrix. Horst (5) has generalized iterations on the correlation matrix for the case $s > 1$, and proof for this generalization has been provided by Householder and Young (8).

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* See also his recent paper, Some new methods in matrix calculation, *Annals Math. Statistics,* 1943, 14, 1-34, especially part III.

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INTERRELATIONSHIPS OF NUMBER-CORRECT AND LIMEN SCORES FOR AN AMOUNT-LIMIT TEST*

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For an amount-limit test homogeneous as to content and varied as to difficulty it is established that an individual's number-right score and his limen score as estimated by the constant process are mathematically related. The experimental and the theoretic relationship between normal deviate and limen score are shown to be in good agreement. It is also found that the two methods of evaluating individual test performance yield equally reliable sets of scores for the procedures used. Accordingly where the assumptions basic to the relationship obtain, the more conveniently computed raw score may be considered to be as valid and reliable an index of individual test performance as the limen score. The concept of the dispersion parameter of the individual as a measure of change or error in test score found no experimental verification. Estimates of individual variability are unrelated to differences in score on equivalent forms.

I. *Introduction*

The evaluation of individual test performance by the analogue of the method of constant stimuli and the related constant process was probably first proposed by Thurstone (10). Mosier (7) further developed the underlying rationale and pointed out several implications for test theory of parameters estimated by means of the constant process. The precise relationship between the constant method evaluation or limen score and the conventional linear evaluation of individual test performance, however, remains to be clarified. Likewise the related question of the relative reliability of the two indices is yet to be investigated. The objectives of this study are then (1) to relate mathematically the conventional number-correct score and the corresponding limen score of an individual for an amount-limit test and (2) to compare empirically with regard to reliability number-correct scores and limen scores.

In order to illustrate the applicability of the method of constant stimuli to mental test data, the usual procedures will be briefly reviewed. For example, in determining a sensory threshold such as the two-point tactual limen, an appropriate range of stimuli judged "two-

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point" neither one hundred nor zero per cent of the time is selected. Each stimulus is administered a large number of times, in a prearranged order, to the observer who is instructed to judge the experience as "two" or "one." The responses are then classified and the relative frequencies of "two" for each grade of the stimulus scale are computed. The stimulus-distance that elicits a response of "two" 50 per cent of the time is conventionally regarded as the threshold. The dispersion of responses in the "two" category constitutes an inverse measure of the precision of sensory discrimination. Both parameters may, of course, be estimated by the constant process.

Now consider, instead of a range of sensory stimuli, a representative range of test items graded in terms of difficulty for a standard group. The items as stimuli are such that the response elicited is classed as either "correct" or "incorrect." Each item may be regarded as having a characteristic response-value that differentiates correct from incorrect responses and thus each item requires for a correct response a given degree of ability as expressed in terms of the standard group. This response-value corresponds to the stimulus-value of psychophysical discrimination. Theoretically if an individual were presented with these items, ordered as to difficulty, and if his responses were made without error, correct responses would be made up to a certain point on the scale depending upon his ability and incorrect responses would be made thereafter. This assumes, in addition to the absence of error, that the individual reacts characteristically with a given degree of ability to all items of a certain type. That is to say, it assumes that there exists for each person a characteristic response-value called the threshold which is the same for all items of a class. Actually, of course, as with sensory thresholds, no sharp dichotomy exists between the two possible types of responses. In actual practice mixed correct and incorrect responses are made to the graded items. Such irregularity of performance is to a considerable extent a consequence of the lack of perfect correlation between items because of error. Hence the psychological process elicited upon presentation of a given item will result in a response which is correct or incorrect depending upon whether the ability or constant component plus the chance error is greater or less than the level of ability required to answer the items correctly. If the composite process is greater, the response is correct; if it is less, the response is incorrect.

As the successive sets of items of increasing order of difficulty are presented to an individual, a progression of proportions p_1, p_2, $\cdots$, p_n of correct response is obtained. If the range of item difficulty is great enough the proportions will run from 1.00 to 0.00 and, if

plotted against the midpoints $x_1, x_2, \cdots, x_n$ of the corresponding difficulty intervals, will tend to give an ogive curve. It is usually found that the integral of the normal curve may be taken as representing this function of item difficulty. The next step would then be to fit the integral of the normal curve to the observed data. The threshold, which is not an observed value but must be estimated from all the data, may be defined as that difficulty-value of the items where the probability is one-half that the individual's response will be correct or incorrect. It represents the characteristic response-value or limen score of an individual. The standard deviation of the ogive function will be a measure of the individual's variability or error. Just as the precision of sensory discrimination varies from individual to individual with respect to a set of stimuli, so does test variability differ from one individual to the next with respect to the sample of items. Thus an individual's goodness of performance may be evaluated and described by means of methods and parameters ordinarily applicable to psychophysical data.

II. *Theoretical Relationships*

The problem now is to relate mathematically an individual's number correct score and his limen score as estimated by means of the constant process for an amount-limit test. The customary procedure in evaluating an individual's goodness of performance on an amount-limit test of n multiple-choice items is simply to sum the number of items answered correctly. Let $X_1, X_2, \cdots, X_n$ be the scores on the individual items; they will be either zero or unity. The test score of an individual is then given by

$$\Sigma X = X_1 + X_2 + \cdots + X_n, \tag{1}$$

where ΣX is the total number of items answered correctly. To correct for chance success, which ordinarily is not infinitely small, this linear sum may be expressed as

$$C = \frac{a}{a-1}\Sigma X - \frac{1}{a-1}n, \tag{2}$$

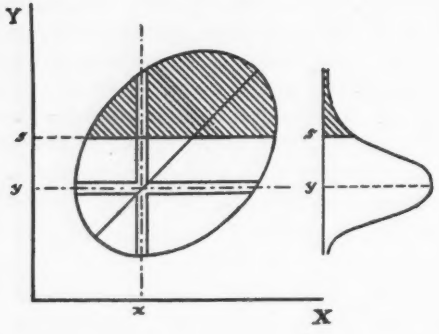
where C is the corrected score, ΣX is the raw score, n is the total number of items, and a is the number of alternatives. It is evident that if every item were attempted by each individual the proportion p_c of all items passed and corrected for chance success, would serve equally well as an index of test performance.

This fact is mentioned because the objective is to relate any person's number-right score to his limen score on a test in which all items

are attempted by every person. It is also evident that p_c, the total proportion of items passed, will be a linear function of C, the corrected total score.

Consider now a normal bivariate distribution with linear regression, and normal and homoscedastic arrays, as represented in hypothetical terms in Figure 1. Let each of the variables X and Y be

FIGURE 1



measured in standard units from its mean of zero. Variable X, which is quantitatively measured and continuous, represents item difficulty or the standard population response to the test items. Variable Y, which is dichotomized into two categories, correct and incorrect, expresses an individual's responses in standard units to the same items. The bivariate distribution thus represents the hypothetical correlation between the responses of a standard group and the responses of an individual to a common set of test items. This formulation, of course, assumes that the trait represented by the dichotomous distribution is actually a continuous trait, normal in distribution, for which we have only categorical information. The normal correlation surface assumed here for our theoretical development implies that each variable is normally distributed (5). In actual practice it is likely that such a restriction need not be placed on the shape of the continuous variable.

Let s, measured in standard deviation units from the mean of Y, be taken as the point of dichotomy of the variable into two categories. It is a measure of the individual's goodness of performance that corresponds to p_c, the corrected per cent of all items passed, and may be secured by recourse to the Kelly-Wood Tables of the integral of the normal curve. In Figure 1 the vertical column represents

any array in the bivariate distribution. The distribution of scores on the Y variable in this array is shown in the normal curve to the right of the ellipse. It represents the frequency or third dimension of the horizontal slice through the bivariate surface. In this array the theoretical proportion of all responses of an individual to items of a given difficulty-value x that are scored as correct lies above s. This is indicated by the cross-hatched area of the normal curve. This proportion varies with x, and may be expressed as a function of x and related parameters.

Let the standard score in Y be designated by y. Then the value of y corresponding to each value of x is given by the regression equation

$$y = r_{xy}\, x\,, \tag{3}$$

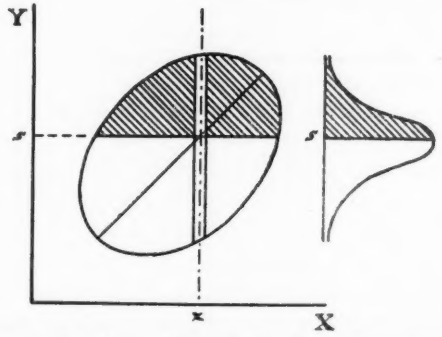
where r_{xy}, the correlation between x and y, is the regression coefficient of y on x. For the vertical x array shown, the corresponding predicted Y value is y, the mean of the distribution of the array. It is shown by the horizontal strip in Figure 1.

The value of s, the individual's standard score, expressed in terms of the standard deviation of the x array, is

$$a = \frac{s - y}{\sigma_{y.x}} = \frac{s - r_{xy}\, x}{\sigma_{y.x}}, \tag{4}$$

where $\sigma_{y.x}$ is the standard error of estimate, and a is the distance between s and y. There will be one array in which $a = 0$ and the median array score will just equal s. This will occur when $s = r_{xy}x$. In this array, which is represented in Figure 2 by a vertical column

FIGURE 2



through the point of intersection of the line of regression with s, the median x array score just equals s. The difficulty value corresponding to this array is then

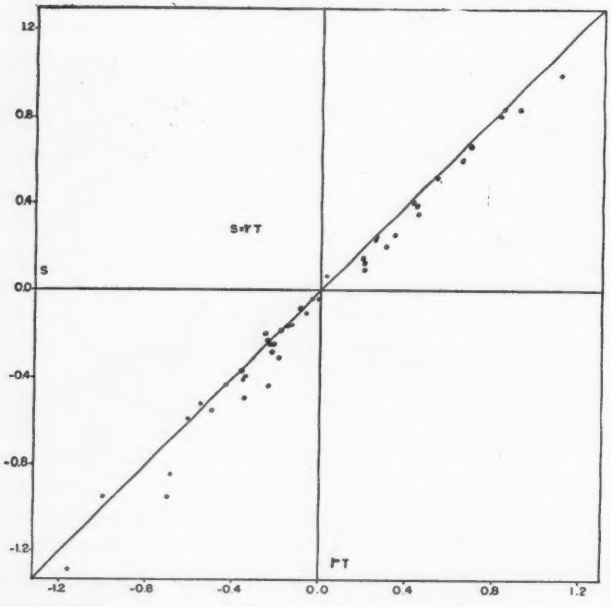
$$t = \frac{s}{r_{zy}}, \tag{5}$$

where t designates the x-value of the array in which theoretically 50 per cent of the responses exceed s and 50 per cent fall below it, i.e., the responses are equally often correct and incorrect.

In agreement with the assumptions adopted, there will exist for each successive array $x_1, x_2, \cdots, x_n$ a theoretical proportion exceeding the value s. Although the observed proportions will differ from the theoretical proportions, it can be assumed consistently with the theory thus far applied, that the integral of the normal curve may be used as the theoretical curve to be fitted. The value t, the threshold, is thus seen to be identical with the threshold value estimated by applying the constant process to the proportions of correct responses secured by the method of constant stimuli. The individual's number correct score on an amount-limit test, if expressed as a normal deviate, is thus directly related to his limen score as estimated by the constant process.

In order to demonstrate experimentally the relation between an

FIGURE 3



RELATION BETWEEN THRESHOLD AND DEVIATE SCORE

individual's limen score and his normal deviate score as related by (5), the plot shown in Figure 3 and the values recorded in Table 1 are presented. The statistics given are C, the corrected raw score; p_c, the obtained proportion of all responses correct; ε_i, the dispersion parameter for an individual; r_{xy}, the correlation between the responses of the standard group and the responses of the individual to the same items; t, the threshold; s, the normal deviate score corresponding to p_c; and rt, the cross-product.

By procedures more fully described and illustrated in the next section, each hypothetical individual was scored as to the number and proportion of his correct responses to six subtests of 112 items that increase in order of difficulty. The proportions, after correction for chance success, were plotted against the midpoints of the median difficulty values of the six subtests on arithmetic probability paper. The individual discrimination curves were fitted by inspection, and estimates of the threshold and the dispersion parameter were secured. The correlations r_{xy} were estimated by means of the relation

$$r_{xy} = \sqrt{\frac{1}{1 + \varepsilon^2_i}}, \tag{6}$$

where ε_i is the obtained dispersion parameter for the individual. Richardson (8) has shown that coefficients thus computed are fairly close approximations to the more usual biserial coefficients.

The figure and the table show the close agreement between the theoretical and obtained values for the relation between threshold score and corresponding proportion correct score expressed as normal deviate. The theoretical values are represented by the solid line. These results suggest that given an amount-limit test homogeneous as to content and varied as to difficulty, the more conveniently computed raw score would represent the individual's performance just as validly as the more logically justifiable threshold score. The plot also serves to explain the correlation of 1.01 between limen scores and number-right scores secured by Mosier (7) after correction for attenuation. It is clear that the two sets of scores bear a proportionate relation to each other.

III. *Reliability Data*

1. *Methods and Procedures.*—The *Cooperative Vocabulary Test, Form Q*, was used for the reliability comparison of the two indices of individual test performance. The test is composed of 210 five-choice items originally selected for inclusion upon the basis of their differentiation of the upper and lower 27 per cent of a standardization pop-

TABLE I

Parameters of the Individual Discrimination Function

Subject No.	C	P_c	ε_i	r_{xy}	t	s	rt
1	126	56	1.06	.68	.28	.15	.19
3	68	30	1.02	.70	− .80	− .52	− .56
5	189	84	1.74	.50	2.00	.99	1.00
6	178	80	1.48	.56	1.50	.84	.84
8	179	80	1.00	.70	1.30	.84	.91
10	22	10	.74	.80	−1.43	−1.28	−1.16
11	98	44	1.18	.64	− .20	− .15	− .13
15	38	17	.98	.71	−1.00	− .95	− .71
16	37	17	.84	.76	−1.32	− .95	−1.00
18	45	20	2.42	.38	−1.80	− .84	− .68
20	62	28	1.19	.64	− .95	− .58	− .61
25	65	29	1.82	.48	−1.05	− .55	− .50
26	75	33	.99	.71	− .60	− .44	− .43
28	80	36	.95	.72	− .50	− .36	− .36
30	89	40	1.25	.62	− .37	− .25	− .23
31	94	42	1.03	.69	− .37	− .20	− .26
32	99	44	1.02	.70	− .20	− .15	− .14
33	92	41	1.04	.69	− .35	− .23	− .24
42	109	49	1.13	.66	.04	− .03	.03
54	115	51	1.45	.57	− .02	.03	− .01
56	135	60	.97	.71	.37	.25	.26
57	119	53	1.19	.64	.03	.07	.02
58	121	54	1.43	.57	.35	.10	.20
61	124	55	1.14	.65	.31	.13	.20
63	135	60	1.21	.64	.51	.25	.33
64	129	58	1.32	.60	.50	.20	.30
67	70	31	.96	.72	− .47	− .50	− .35
69	75	33	1.75	.50	− .46	− .44	− .23
70	77	34	1.71	.51	− .71	− .41	− .36
72	78	35	1.24	.63	− .55	− .39	− .35
74	85	38	.91	.74	− .25	− .31	− .19
75	88	39	1.00	.70	− .31	− .28	− .22
76	89	40	.90	.74	− .31	− .25	− .23
78	97	43	1.05	.68	− .26	− .18	− .18
80	132	59	.99	.71	.35	.23	.25
85	144	64	.98	.71	.63	.36	.45
86	146	65	1.07	.68	.65	.39	.44
87	147	66	.97	.71	.59	.41	.42
92	157	70	.88	.75	.70	.52	.53
94	105	47	.64	.84	− .11	− .08	− .09
96	102	46	1.16	.63	− .11	− .10	− .07
100	164	73	1.09	.67	.97	.61	.65
104	177	79	.96	.72	1.21	.81	.82
105	167	75	.86	.76	.90	.67	.68

ulation. The test was administered to 500 Washington, D. C., high-school seniors. Approximately an hour was required for all persons to complete the entire test.

The 500 answer sheets were machine-scored and the total raw scores obtained were corrected for chance success. The answer-sheets were then arranged in alphabetical order and a random sample of 100 was selected to constitute the experimental group. To transform the total raw scores into normalized standard scores and to subdivide the transformed scores into a suitable number of class intervals, the pro-procedure was as follows. The 400 standard population answer-sheets were arranged in order of size of score and split into groups of 50 each. The total score of the 50 subjects was considered to be a centroid given by

$$\bar{x} = \frac{z_1 - z_2}{q}, \tag{7}$$

where $\bar{x}$ is the centroid or mean deviation of a portion of a unit normal distribution included between the ordinates z_1 and z_2 which enclose the segment at the left and right, respectively; and q is the proportion of cases lying in the sector $z_1 - z_2$ (4).

The difficulty of an item was ascertained by first securing a graphic item count of the number of correct responses to the item of the 50 subjects in each of the eight categories. The eight percentages for each item (corrected for chance success) were then plotted on arithmetic probability paper against the previously computed centroid values, and a straight line was fitted by inspection. The difficulty of an item was taken as the standard score of the total test at which the item was equally often passed or failed (8). The slopes of the resulting item curves were used as a rough index of discriminatory validity. Items presenting flat curves were eliminated from the experimental test. The inclusion of the rejected items in the total scores used for computing item difficulty was assumed to have a negligible effect principally because of the width of the class intervals used in classifying the items.

On the assumption that difficulty overcome is normally distributed, the experimental items were distributed normally into six equal class intervals. Each interval, 0.74 sigma units in width, was assigned an x-value equal to the midpoint of the class interval. The 112 items culled from the original 210 were split into two equivalent experimental scales or forms of 56 items each.

The number and theoretical per cent of the items in each interval for forms A and B are given in Table 2.

TABLE 2
Distribution of Items in the Difficulty Intervals

Midpoint of class interval in standard units	Per cent in interval	Number in each interval
−1.85	6.8	4
−1.11	16.7	9
−0.37	27.0	15
0.37	27.0	15
1.11	16.7	9
1.85	6.8	4

The midpoints of the tail intervals of the distribution were taken as 1.85 and −1.85. Although these are not the correct values, the error introduced is probably negligible.

Since the numbers of easy and difficult items in the difficulty scale are few, some means had to be devised to insure the stability of response to items in these tail intervals. By combining subjects on the basis of total score on the same test form into hypothetical individuals, this stability of response can be secured. Accordingly, an additional 200 subjects were given the complete form. The answer-sheets were then scored on the 112 experimental items. Of this group one hundred were paired with the original experimental population on the basis of total raw score. Each pair considered as a hypothetical individual was scored on both test forms for total number of correct responses and the obtained values were corrected for chance success.

The next procedure was to calculate for each hypothetical person the number and proportion of his correct responses to the items in each difficulty interval on the two experimental forms A and B. The obtained proportions, after correction for chance success (3), were plotted on arithmetic probability paper as the ordinate values against the midpoints of the corresponding difficulty intervals as abscissae. The abscissae were those shown in Table 2; they represent the difficulty values of the items for the standard group and are expressed in standard units. The easier items are negative in sign and the more difficult items are positive in sign. The ordinates, similarly, were in standard units although the values listed on arithmetic probability paper along the vertical axis are percentages. Naturally the greater the proportion of items passed by an individual, the more capable the individual. It has been assumed here that the normal ogive will represent the relation between these proportions and the difficulty values of the corresponding items.

The individual curves were fitted by inspection and graphic smoothing, and estimates of the limen and the standard deviation parameters were secured for each hypothetical individual for both experimental test forms as well as for the combined form of 112 items. The limen was taken to be that difficulty value of the scale where the items were equally often passed and failed by the hypothetical individual. The dispersion parameter was estimated by determining the two points of the difficulty scale at which 84 and 50 per cent (or 50 and 16 per cent) of the items were passed.

To determine the stability of the method of fitting as opposed to the constant process, eight curves were fitted by the Müller-Urban procedure as outlined in Guilford (2). The differences were found to be slight enough to outweigh the laboriousness of the more preferable procedure. Application of the constant process to secure estimates from all the data would have been, no doubt, more admirable theoretically. Yet under practically all conditions except theoretic, other methods are preferable. It is seldom that the data warrant or

TABLE 3
Distribution of Limens and Corrected Scores of Forms *A* and *B*

Limen Scores				Corrected Scores			
Class Intervals		Frequencies Form *A*	Frequencies Form *B*	Class Intervals		Frequencies Form *A*	Frequencies Form *B*
−1.99	−1.80	1	0	5	9	1	1
−1.79	−1.60	0	0	10	14	4	0
−1.59	−1.40	3	3	15	19	3	4
−1.39	−1.20	1	3	20	24	2	4
−1.19	−1.00	1	3	25	29	3	5
− .99	− .80	6	4	30	34	6	1
− .79	− .60	9	10	35	39	6	3
− .59	− .40	7	6	40	44	9	8
− .39	− .20	12	10	45	49	8	6
− .19	− .00	4	11	50	54	10	14
.01	.20	13	14	55	59	7	10
.21	.40	12	14	60	64	11	8
.41	.60	8	3	65	69	8	9
.61	.80	8	1	70	74	6	10
.81	1.00	3	3	75	79	4	5
1.01	1.20	2	6	80	84	4	3
1.21	1.40	4	1	85	89	4	2
1.41	1.60	3	1	90	94	3	4
1.61	1.80	0	2	95	99	1	3
1.81	2.00	3	3				
2.01	2.20	0	1				
2.21	2.40	0	1				

require greater accuracy (1). A clinician, for example, will have neither the time nor the inclination to compute an Urban limen. The graphic method on the other hand provides greater accuracy than that given by linear interpolation. Of course, if the plot is not rectifiable the data are probably not worth application of the more exacting procedure—or the graphic methods for that matter.

2. *Test Results.*—The two sets of number-correct scores and the two sets of threshold values for the one hundred experimental subjects were correlated to secure estimates of their respective reliabilities. The distribution of raw scores corrected for chance success and the corresponding thresholds are presented in Table 3 for forms A and B. The number-correct scores on forms A and B correlated .887, while the limen values correlated .897. The Spearman-Brown formula for double length yielded estimates of .94 and .945, respectively, for the two methods of scoring the experimental tests. The comparability of the two experimental forms may be judged by the data given in the table below:

TABLE 4
Means and Standard Deviations for Forms *A* and *B*

	Mean Score	Standard Deviation	Mean Limen	Standard Deviation
Form *A*	53.3	20.8	0.07	0.79
Form *B*	56.4	19.9	0.04	0.83

These results differ somewhat from those found by Mosier (7) for a larger group of heterogeneous items and a smaller population (80 subjects). The corrected reliability of the limen scores based on odd-even items was .88 compared with a value of .94 for the number-correct scores. The findings for this study, on the other hand, indicate that for the assumptions and the procedures here used the two methods of scoring yield equally reliable sets of scores. This conclusion is consistent with the theory on which the proportionality relationship between s and t is based.

The reliability of a test, of course, represents an average for a particular set of individuals. Similarly the standard error indicates the average change in any person's test score as the test is replaced by an equivalent form with the same mean, standard deviation, and reliability coefficient. Actually there is no reason to suppose that the error of a test score is a constant, that is to say, the same for all individuals for a given test. It would appear, therefore, that ε_i as obtained by the method of constant stimuli represents a measure of the change or error in an individual test score as successive samples of

test items are drawn from the same universe of test items. Mosier (6) in particular has developed the concept of the standard deviation of errors for a particular individual in a test of items.

Accordingly an attempt was made to relate ε_i as a psychological characteristic of the individual to (1) the absolute and algebraic change in limen score from one test form to the next and (2) to the limen score. The value of ε_i for each of one hundred individuals was computed graphically on arithmetic probability paper from the obtained proportions of the correct responses on the combined experimental forms. The computed values of ε_i were then plotted against the difference $t_a - t_b$, where t_a and t_b represent the thresholds on the two test forms for single individuals. No relation could be observed either for the absolute differences or the algebraic differences, as plotted against the ε_i values. These results would indicate that the concept of ε_i as a measure of change or error in an individual's score is in need of empirical verification or clarification.

The measures of ε_i were also plotted against the limen scores of each subject. Again no relationships were discernible. Here it was supposed that there might be variation of the dispersion parameter for the individual with the absolute magnitude of score.

Since the dispersion parameter of a hypothetical individual is of doubtful psychological significance, its stability is probably not worth investigating. On the other hand, the constancy of ε_i for an actual subject could not be examined satisfactorily because of the relatively small numbers of items in the two difficulty scales. When estimates of the dispersion parameter were computed for one hundred actual subjects on both scales by procedures previously described, the equivalent form reliability was found to be only .47.

IV. *Summary and Conclusions*

(1) It is established for an amount-limit test of varied difficulty that an individual's number-correct score and his limen score, as estimated by the constant process, are mathematically related. The empiric relation between normal deviate score and limen score is shown to be in good agreement with the theoretic relationship. This indicates that the assumptions basic to the rationale are reasonably well justified. The findings also imply that where the requirements of the relationship are met the more conveniently computed raw score is as valid an index of individual test performance as the limen score.

(2) For the assumptions made and the procedures used it is found that the two methods of evaluating individual test performance yield equally reliable sets of scores. This conclusion is consistent with the established relationship existing between threshold and raw score.

Accordingly the conventional linear evaluation of test performance may be used with added confidence.

(3) The concept of the dispersion parameter of the individual as a measure of change or error in test score found no verification in the data used to test its plausibility. It may be that the estimates of individual variability are related to retest changes in score rather than to differences in score on equivalent forms. As is well known, test-retest reliability differs from equivalent form reliability.

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THE APPLICATION OF PROBIT ANALYSIS TO THE RESULTS OF MENTAL TESTS

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The application of the Müller-Urban constant process to item selection, as considered in a recent paper in this journal, is shown to be closely analogous to a method now in general use for the analysis of insecticidal and other toxicological tests. This method of *probit analysis* gives the maximum likelihood estimates of the unknown parameters and, with the aid of published tables, the necessary computations can be rapidly performed. The present paper contains an outline of the method and an illustration of the most convenient form of computations for use in analyses of psychometric data.

1. *Introduction*

Methods for the estimation of the limina and precisions of separate items of a mental test have recently been discussed by Ferguson (1). In such a test each item admits of only a quantal response from each person attempting it: either he passes or he fails. The problem considered by Ferguson is thus closely analogous to those arising in connection with other material giving quantal responses, and in particular with much toxicological experimentation, in which insects or other test organisms are exposed to poisons and recorded as having died or survived.

The statistical analysis of toxicological and other dosage-mortality data has received much attention in recent years. The chief development has been the method of *probits*, which, though owing something to earlier writers, arose primarily from the papers of Gaddum (2) and Bliss (3-6). Tables originally used by Bliss for the numerical analysis have been given by Fisher and Yates (7), together with a concise account of the arithmetical procedure. The present paper illustrates the application of the probit method to the data examined by Ferguson.

2. *The Probit Method*

When the problem of the toxicologist is compared with the psychometrical problem discussed by Ferguson, it is seen that the analogue of the *dosage* of poison is the *ability* of a group of subjects, and the analogue of the *poison* itself is the *item* under investigation. The toxicologist measures the effectiveness of the dosage by the propor-

tion of test organisms which succumb to it, the psychometrician measures the difficulty of the item by the proportion of subjects who fail.

If the measure of ability, x, can be so chosen that the proportion of persons of ability x_0 in the whole population studied who could pass the item is given by the normal integral

$$P = \int_{-\infty}^{x_0} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}\, dx,$$

μ may be described as the *limen* of the item, the level of ability at which 50 per cent of subjects will pass. This implies that the proportion of persons of ability x who are just on the threshold of passing the item is normally distributed, σ being the standard deviation of this normal distribution.

The *probit* of a proportion P has been defined (5) as the value of Y satisfying the equation

$$P = \int_{-\infty}^{Y-5} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du; \qquad (1)$$

Y is the unit normal deviate corresponding to P increased by 5, this being a conventional addition in order to avoid the occurrence of negative values. The probit of P is thus linearly related to x by the equation

$$Y = 5 + \frac{1}{\sigma}(x - \mu). \qquad (2)$$

The problem of estimating the parameters μ, σ is equivalent to the problem of finding the best fitting linear relationship between ability and the probit of the proportion passing the item. This has been recognized by Ferguson and is the basis of the "constant process" used by him. Lawley (8) has considered the application of Fisher's method of maximum likelihood (9) to the estimation of the parameters, without knowing of the methods of Gaddum, Bliss, and others now in general use in the analysis of toxicological data, and he has obtained an approximate solution of the problem. A difficulty arises when percentages passing in any one group of subjects are very low or very high, or when, on account of small numbers of subjects or for other reasons, the data are very irregular; the methods of Ferguson and Lawley may then be misleading or inapplicable. Perhaps the most elegant form of analysis derives from Fisher's solution of the maximum likelihood equations (Appendix to 5), which has been discussed in greater detail by Garwood (10). Fisher and Yates, in

the introduction to their *Statistical Tables* (7), have shown the form most conveniently used for the computations.

The first step is to plot the probits of the observed percentages ("empirical" probits) against ability, x; the probits of 0 and 100 per cent are infinite and cannot be shown on this diagram. A straight line is then fitted by eye to these points, making rough allowance for any points at infinity. With experience, and with regular data, it is possible so to draw this line that both the limen and the standard deviation may be estimated satisfactorily from it, though it is generally used only in order to obtain a better approximation to the maximum likelihood solution.

"Provisional probits," Y, are read from this line for each value of x in the data, and from these are derived "working probits," y, where

$$y = Y + \frac{Q}{Z} - \frac{q}{Z}.$$

In this equation $Q = 1 - P$, where P is related to Y by equation (1) and Z is the ordinate to the normal curve,

$$Z = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(Y-5)^2};$$

$q(= 1 - p)$ is the observed proportion failing the item amongst the subjects with this value of x. The working probit remains finite when $q = 0$ or 1 and the empirical probit is consequently infinite. For other values of q, if the empirical probit is close to the provisional line, the working probit has practically the same value. Use of the working probit serves to eliminate a bias, arising from the asymmetry of the binomial probability law, which is more important the greater the divergence of the empirical probit from the provisional line.

An improved estimate of the relationship between x and the proportion passing is then given by forming the linear regression equation of y on x, giving each y the weight $nw = nZ^2/PQ$, where n is the number of subjects of ability x. This regression equation takes the form,

$$Y' = \bar{y} + b(x - \bar{x}),$$

$\bar{x}$, $\bar{y}$ being the weighted means of x and y, and b the regression coefficient. If this line differs considerably from the provisional line, the process may be repeated using Y' instead of Y, but usually this last equation will give the maximum likelihood estimates of the parameters sufficiently closely as

$$m = \bar{x} + (5 - \bar{y})/b$$
$$s = 1/b\,. \tag{3}$$

The numbers of subjects in each group who pass the item may be compared with the numbers predicted by the values of Y', a χ^2 test being used to test the significance of the discrepancies. If the agreement is considered satisfactory, standard errors of the estimates of the two parameters and of associated quantities may be derived as illustrated in the next section.

3. *Application to Ferguson's Data*

In toxicological applications of the probit method the measurement of dosage used is either the quantity or concentration of the poison or, more commonly, the logarithm of the concentration. When investigating the psychometrical problem, Ferguson had a further difficulty in that he had no measure of the "ability" of his subjects independently of his data. He overcame this ingeniously by using the total score for the whole test to obtain such a measure. This score includes the results for a particular item, and thus is not independent of the item, but is open to little objection when the test consists of a large number of items so that the passing of one has little influence on the total score. Ferguson assumed that the distribution of "ability" is normal, and he therefore took as x_A, the measure of ability of a subject A, the normal equivalent deviation of the subject's score, defined as the deviation of the unit normal distribution corresponding to the percentage of subjects having a lower score than A's. Thus x is subject to errors of estimation, which should not be serious in a large sample.

For convenience, the 216 subjects were grouped into seven groups, in intervals of 0.6 of x. Ferguson took the arithmetic mean of the end points of each interval as the x-value for the group, but it seems preferable to take the mean according to the frequency distribution; this may be shown to be

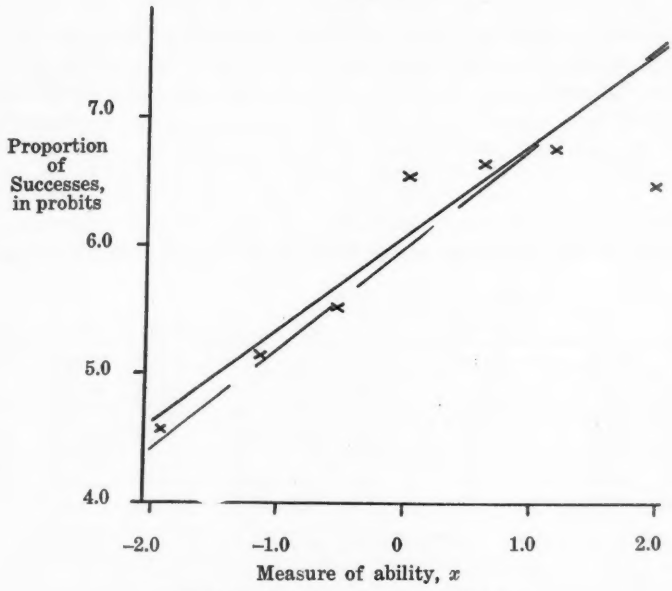
$$x = (Z_1 - Z_2)/(P_2 - P_1),$$

where P, Z are the area of the normal distribution to the left of the end-point of the interval and the ordinate at the point, respectively, and may be obtained from tables (7, Tables IX and II). For example, for the interval from 0.3 to 0.9, the area to the left of $x = 0.3$ is, from Table IX, 0.6179, and the ordinate at the point 0.3814; values are similarly obtained for $x = 0.9$. Hence the mean value of x for the interval is

$$(0.3814 - 0.2661)/(0.8160 - 0.6179) = 0.582 .$$

The computations for estimating the parameters for Item 1 of Ferguson's paper are given in Table 1. The first four columns show x, the number of subjects in each group, n, the proportion of these who pass (as given in Ferguson's Table 2), p, and the probit of p, read from Fisher and Yates' Table IX. These empirical probits are plotted against x in Fig. 1 and a provisional line drawn through the

FIGURE 1



Probit Diagram for Data of Item 1.
x empirical probits
--- Provisional regression line, Y
—— Fitted regression line, Y'

seven points, rough allowance being made for the much greater weight to be attached to the points with probits near to 5. The limen may be estimated directly from this line as the value of x for which the probit is 5; this gives

$$m = -1.32 .$$

Likewise s may be estimated as the reciprocal of the slope of the line.

In order to proceed with the maximum likelihood estimation, values of the provisional probit, Y, are read from the line and entered

in Table 1; one decimal place is generally sufficient for Y. Fisher and Yates' Table IX is then used to give the weights, nw, and working probits, y. For example, for the highest level of ability $Y = 7.5$; there are 15 subjects in this group and therefore

$$nw = 15 \times 0.0498 = 0.7 .$$

Also, the working probit is

$$y = 7.8543 - (1 - 0.93) \times 57.05 = 3.86 .$$

This is the only one of the seven working probits that is appreciably different from the corresponding empirical probit. When the provisional line is a good fit to *all* points, there is little necessity to replace the empirical by the working probits, but where there is a serious discrepancy the working probit must be used in order to allow the ob-

TABLE 1

Computations for the Calculation of the Probit Regression Equation for Item 1

x	n	p	Empirical probit	Y	nw	y	nwx	nwy	Y'
−1.94	15	0.33	4.56	4.5	8.7	4.56	−16.878	39.672	4.66
−1.16	25	0.56	5.15	5.1	15.9	5.15	−18.444	81.885	5.23
−0.58	43	0.70	5.52	5.5	25.0	5.52	−14.500	138.000	5.65
0.00	50	0.94	6.55	6.0	21.9	6.41	0.000	140.379	6.08
0.58	43	0.95	6.64	6.5	11.6	6.63	6.728	76.908	6.50
1.16	25	0.96	6.75	6.9	3.9	6.73	4.524	26.247	6.93
1.94	15	0.93	6.48	7.5	0.7	3.86	1.358	2.702	7.50
					87.7		−37.212	505.793	

$Snwx^2$	$Snwxy$	$Snwy^2$
+74.33296	−171.69524	+2961.17353
−15.78943	+214.61310	−2917.06453
$S_{xx} = 58.54353$	$S_{xy} = 42.91786$	$S_{yy} = 44.10900$
		31.463
		$12.646 = \chi^2_{[5]}$

$$\bar{x} = -37.212/87.7 = -0.4243$$
$$\bar{y} = 505.793/87.7 = 5.7673$$
$$b = 42.9179/58.5435 = 0.7331$$
$$Y' = 5.7673 + 0.7331\,(x + 0.4243)$$
$$= 6.078 + 0.733\,x$$

servation to exercise its correct influence on the results; it is therefore preferable to adopt the fully correct method as a routine, rather than to consider each case on its own merits.

The remainder of Table 1 shows the determination of the regression equation of y on x. The equation is

$$Y' = 6.078 + 0.733\,x, \qquad (4)$$

and the values of Y' are given in the last column of the table. The computations could then be repeated using Y' as the provisional probit, but the agreement between this line and the provisional, as shown on Fig. 1, is so good as to dispose of any necessity for this*.

Using S_{xx}, S_{xy}, S_{yy} to denote the sums of squares and products of deviations of x and y about their means, the expression

$$S_{yy} - S^2_{xy}/S_{xx}$$

may be shown to be equivalent to the χ^2 value attained by comparing the observed numbers passing and failing, np and nq, with the expected numbers, nP' and nQ', where P' is the proportion corresponding to the probit value Y'. As is shown in Table 1, this gives

$$\chi^2 = 12.646,$$

with degrees of freedom 5, this being two less than the number of groups. Reference to tables of the significance levels of χ^2 (Fisher and Yates, Table IV) indicates that this value is significant, implying either that the subjects of a group do not give independent results, or that a straight line is inadequate to express the relationship between probit and ability.

This method of obtaining χ^2 is rapid and in many cases sufficiently accurate, but it tends to overestimate the value when the expected numbers passing or failing in some groups are very small. If χ^2 does not exceed the significance level, it is safe to assume that the data are in satisfactory agreement with the fitted line, but when χ^2 is large it is often found that much of it arises from one or two groups in which the expected numbers are very small. It is well known that, in the latter case, the χ^2 test is unreliable unless some groups are amalgamated and larger expectations obtained. In the present example, closer examination shows the large value of χ^2 to arise chiefly from the highest ability group, in which 14 subjects passed as compared with 14.9 expected; the expected number failing, 0.1, is thus very small. Amalgamation of the three highest groups, in the manner suggested by Finney (11), removes the unreliability of the test, and

* If values of Y' are rounded to one place of decimals, the next approximation attained is $Y'' = 6.083 + 0.736\,x$, a trivial change from Y'.

gives a χ^2 (with 3 degrees of freedom) well within the limits of random variation.

It then follows that the variances of $\bar{y}$, the mean probit, and b, the regression coefficient, are $1/Snw$ and $1/S_{xx}$, respectively. From the expression for m in equation (3) it may be seen that the variance of the limen is

$$V(m) = \frac{1}{Snw} + \frac{(m - \bar{x})^2}{S_{xx}}$$

$$= \frac{1}{87.7} + \frac{(1.047)^2}{58.54}$$

$$= 0.0301 .$$

Putting $Y' = 5$ in equation (4) gives the new estimate of the limen, and its standard error is the square root of the variance; hence

$$m = -1.471 \pm 0.173 ,$$

a value not greatly different from that read directly from the provisional line.

TABLE 2

Comparison of Estimates Obtained by the Constant Process and the Probit Method

Test item	Limen		Regression coefficient	
	Constant process	Probit method	Constant process	Probit method
1	–1.48	–1.47 ±0.17	0.69	0.73 ±0.13
2	–0.88	–0.87 ±0.13	1.00	1.05 ±0.15
6	–0.03	0.02 ±0.10	0.95	1.09 ±0.12
12	0.06	0.04 ±0.09	0.76	0.76 ±0.11
20	0.64	0.65 ±0.11	0.71	0.73 ±0.11
28	1.79	1.77 ±0.20	0.53	0.56 ±.011

In Table 2 are shown the estimates of limina for the six items as obtained by the probit method, compared with the estimates obtained by Ferguson from the constant process. The differences are trivial for these data, but the probit method, in addition to its theoretical advantages, leads also to values of the standard errors of the limen. The regression coefficients obtained by the two methods are also shown in Table 2, those for the constant process being reciprocals of Ferguson's "standard deviations." Again the probit method also yields standard errors of estimation. The constant process consistently underestimates this parameter, by comparison with the maxi-

mum likelihood values, probably because of zero and 100 per cent passes having to be ignored in the constant process.*

In a short paper it is not possible to consider fully the application of the probit method to this type of data, and extended applications to toxicological material may also find their analogues in the analysis of psychometric tests. For example, when two or more items of a test yield probit regression lines which do not depart significantly from parallelism, the differences between the items may conveniently be expressed in terms of the *mean ability difference*, the horizontal distance between the lines, in the same way as relative potency is used in toxicological work (12). A test for parallelism is obtained by comparing the sum of the residual χ^2 values for the lines when a separate b is fitted to each item with the corresponding value when b is constrained to be the same for all items. Again, if the influence of two separate measures of ability on test results is to be investigated, the regression method easily extends to the concept of the *probit plane* (13), which may be useful in interpreting the data.

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* As in all regression problems, any errors of estimation of the independent variate (here the ability) will also lead to underestimation of the regression coefficient, irrespective of whether the probit method or constant process is used. This effect should not be important when the test consists of a large number of items, as the estimate of ability will then have only a small sampling error.

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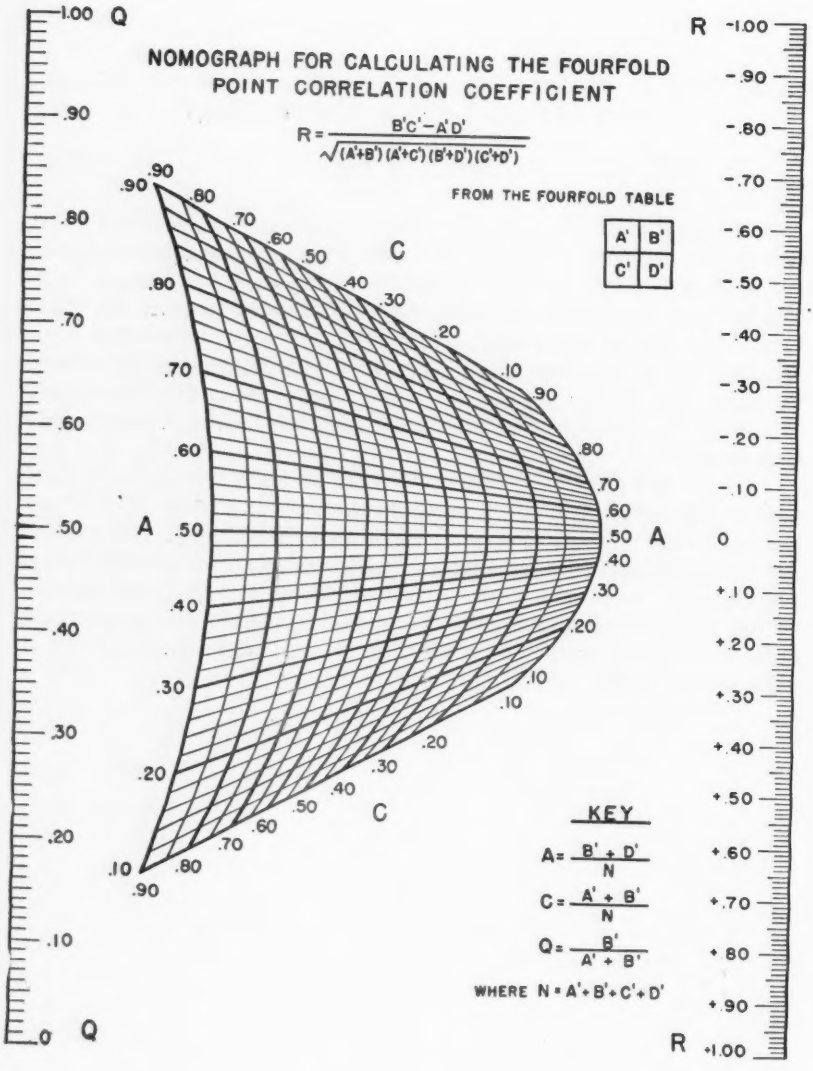
ALIGNMENT CHART FOR CALCULATING THE FOURFOLD POINT CORRELATION COEFFICIENT

FREDERIC M. LORD

The alignment chart may be used to calculate the fourfold point correlation coefficient for any fourfold table. If the categories into which each variable of a fourfold table is classified are termed "successful" and "unsuccessful," then the values needed to enter the chart are: A, the per cent of cases successful with respect to the first variable; C, the per cent of cases successful with respect to the second variable; and Q, the per cent of those cases successful with respect to the second variable that were also successful with respect to the first variable.

To use the alignment chart: Find the straight line in the central part of the chart that corresponds to the desired values of A. Find the intersection of this line with the curved line that corresponds to the desired value of C. By means of a straightedge connect this intersection with the point on the linear scale at the left that corresponds to the desired value of Q. Read the value sought for the fourfold point correlation coefficient where the straightedge intersects the R scale on the right.


NOMOGRAPH FOR CALCULATING THE FOURFOLD POINT CORRELATION COEFFICIENT
$R = \frac{B'C' - A'D'}{\sqrt{(A'+B')(A'+C')(B'+D')(C'+D')}}$
FROM THE FOURFOLD TABLE
A' B'
C' D'
Q
R
A
C
KEY
$A = \frac{B' + D'}{N}$
$C = \frac{A' + B'}{N}$
$Q = \frac{B'}{A' + B'}$
WHERE N = A'+B'+C'+D'

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A SEMI-ANALYTICAL METHOD OF FACTORIAL ROTATION TO SIMPLE STRUCTURE

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A factorial rotational method is presented which represents a compromise between the use of subjective judgment characteristic of graphical methods and the routine application of analytical methods. At present the analytical methods seem to be inadequate for the discovery of a simple structure, while graphical methods require more subjective judgment. The method herein presented locates the axes for subgroups of tests by an analytical method. The judgments used in the selection of subgroups are based on graphic data concerning interrelation of the factors.

During the development of multiple factor analysis the rotation of the reference axes has risen to great importance. The position of the accepted set of axes depends upon the purpose of the individual factorial study, as does also the method of rotation employed. In general, that set of axes should be selected which is most useful in solving the problem for which the study was devised. Thus the principle underlying the method of rotation depends upon the use to which the entire study is to be put. For some problems the principal axes are the most useful; other studies merely demand that the axes be orthogonal; still others require rotation to positions where the underlying nature of the factors is more easily deduced from the saturations of the tests. It was for this last case that Thurstone (3) developed the principle of simple structure. It is the belief of those who employ this principle that axes can be passed through the factorial configuration so that a large number of tests have vanishingly small loadings.

Although the principle of simple structure seems to be clear-cut, the methods of rotation have been cumbersome and difficult to apply. However, sufficient experience with these methods has now been obtained to allow a critical review of their difficulties as well as of their merits and thus to permit the development of a compromise between the routinely applied analytical methods and the more subjective graphical methods.

The most successful rotational methods have been the graphical ones devised by Thurstone (5, 6). The results from these procedures

have, in general, been satisfactory; however, these methods depend upon subjective judgments of the investigator and uniform results are not obtained by all. Successful solutions of rotational problems by these methods depend upon a highly sophisticated sense of geometrical projection and of fitting lines to points. Further, the computational labor seems unnecessarily great, especially when the results are carried to a high degree of precision.

In terms of objectivity of procedure an analytical method of rotation is to be preferred. Identical results could then be obtained by all investigators, and thus the solution could be termed unique. However, analytical procedures have a number of inherent difficulties. Usually, for each test j a value y_j is assigned for the corresponding loading v_j and then the sum of the y's is maximized, or minimized, for each of the rotated axes. Whether a maximum solution or a minimum solution is desired depends upon the definition of the y's. An example of a case where a maximum sum of y's is desired is when

$$y_j = e^{-cv_j^2},$$

and

$$\sum_j y_j = \text{a maximum}.$$

In this case y_j increases when the absolute value of v_j decreases and has its highest value for a v_j of zero. A minimum sum of y's is desired when the lowest value of y_j corresponds to a zero v_j. The important point to note is that all tests are entered undifferentially into the formulation. Thus the position of the axis is influenced not only by those tests which are to have zero or near zero loadings, but also by those with high loadings. This results in a shift in the axis from the desired position. The amount of this shift varies both with the study and with the particular definition of y_j. An exceptionally bad example is seen in the case where y_j is defined as the square of v_j. When the sum of the y's is minimized the result is the mean principal axis of the entire system where all of the test projections are brought to as near zero as possible. Many of the other definitions of y_j tend toward this same result.

Thurstone (4) has suggested, as a possible means of overcoming the inadequacies of analytical criteria, the consideration of a subgroup of tests rather than the entire battery when taking the sum of the y's. This procedure was further developed by Tucker (7), who defined the simple structure factor as the mean principal axis of the zero subgroup of tests. In general, however, the solutions for any assumed subgroup cannot be easily obtained directly and recourse has been made to laborious methods of successive approximation. An-

other difficulty is in the selection of the subgroup to be treated. If the subgroup is taken as those tests which lie within a certain range from zero on a trial vector, the resulting solution is merely a tightening up of the tests in this range and the trial vector could easily remain a combination of two or three of the simple structure factors. This results because all tests high on the two or three factors have loadings on the original trial vector outside of the range taken in defining the subgroup. A more satisfactory method is to solve for all factors simultaneously, selecting subgroups from the inter-factor graphs.

Although the use of subgroups has led to improved results, there still exists a small tendency for the trial factors to rotate toward the mean principal axis of the entire system. This tendency is especially pronounced when the variance on the mean principal axis is low and this axis can be considered as a residual dimension. The major defect of rotation in this direction is that the tests with non-vanishing loadings on the factors have smaller loadings than if this rotation were not carried out. There is a loss of variance of projection for these tests.

A solution for this problem of loss in size of the high projections is possible with the criterion suggested by Horst (1) who maximized the ratio of the sum of squares of the high projections to the sum of squares of the low projections; that is, he divided the tests into two subgroups, one for those tests with non-vanishing loadings and one for those tests whose loadings were to be vanishing. The tests in the high subgroup will be designated here by the subscript h, those in the zero subgroup by the subscript z. Horst's solution was then to make

$$\frac{\sum_h v_h^2}{\sum_z v_z^2} = \text{a maximum.} \qquad (1)$$

This criterion seems to have the proper characteristics when the correct subgroup has been chosen. However, in applying this criterion Horst did not make use of graphical selection of the subgroups, preferring, rather, the use of a range of projections on a previous trial. The method proposed here makes the solution for all factors simultaneously, using the inter-factor graphs as guides for the selection of the subgroups. As a further aid in computation, an approximation to Horst's criterion is used during the trials where the subgroups are being selected.

The fictitious problem used by Thurstone to illustrate rotation with extended vectors (6) will be used here to demonstrate the prop-

erties of the proposed method. Table 1 presents the fictitious simple configuration which was rotated to the centroid matrix of Table 2. In practice the problem is to start with a centroid matrix such as is given in Table 2 and rotate to the simple configuration. In this case, however, it is convenient to have the exact solution given in Table 1 in order to judge the goodness of the rotations.

The first problem in applying the method is that of obtaining an initial set of trial subgroups. These subgroups may be obtained by a cluster analysis of the original correlations, by inspection of the centroid factorial matrix, or, as is suggested here, by a cluster analysis of the first factor residuals. It is assumed that the configuration has not only a simple structure, but also a positive manifold. When this is true, as with most test data, the first centroid axis passes somewhere in the middle of the positive manifold and the test vectors are projected down into the space orthogonal to this axis. Not all the residual correlations are positive. In fact, positive residuals occur only for those pairs of tests that extend in the same general directions and thus have loadings on the same factors.

Table 3 gives the first factor residuals for the illustrative problem. It is noted that the highest residual in the table is between variables 2 and 14. These two tests are taken as the starting point for the first cluster. The next test added to this cluster is test 4, which correlates .45 and .30 with tests 2 and 14, respectively. Succeeding tests are added to the subgroup in such a way that all of the residual intercorrelations are positive. The tests in this cluster are listed in column A of Table 4.

Cluster A accounts for a number of the positive residuals. The next cluster is started by finding the highest residual that has not been accounted for by cluster A. This residual is between variables 1 and 13. Cluster B, of Table 4, is found starting with these tests. All of the residual intercorrelations within this cluster are also positive. The other clusters are found in a similar manner. Since the problem has five dimensions as shown by the five columns of the centroid matrix of Table 2, five clusters are taken. In general, as many clusters should be taken as there are dimensions in the factorial matrix. These clusters will be used as the first assumed subgroups of "high" tests.

There is one other preparatory set of computations to be made before starting on Trial 1. The centroid factorial matrix of Table 2 must be premultiplied by its transpose, thus producing the matrix $\boldsymbol{P}$; that is,

$$\boldsymbol{P} = \boldsymbol{F}'_0 \boldsymbol{F}_0 . \qquad (2)$$

This product matrix for the illustrative problem is given in Table 5. The diagonal entries are the sums of the squares of the corresponding columns of F_0 and the off-diagonal entries are the sums of cross-products between columns of F_0. The inverse of P is also found and is listed in Table 6. P^{-1} will be used in the calculations for each trial.

The computational work for Trial 1 is shown in Table 7. The matrix W has a row for each of the tests and a column for each factor. The factor A in matrix W corresponds to cluster A of Table 4. Each of the tests in this cluster has an entry of unity for this factor. All other cells in the column are left blank to indicate entries of zero. The other columns of W correspond to the other clusters.

The first step in Trial 1 is to find the matrix B by the equation:

$$B = F'_0 W . \tag{3}$$

The matrix L is then found by:

$$L = P^{-1} B . \tag{4}$$

The columns of L are then normalized to the corresponding columns of Λ. The matrix C is found by:

$$C = \Lambda' \Lambda . \tag{5}$$

This matrix shows the cosines of the angles between the trial factors and should be closely inspected to see if any pair of the factors are nearly identical. If this should occur, a new cluster should be tried. This is shown by a high positive side entry. The diagonal entries should be unity if the normalizing has been done properly. The last computational step of the trial is the finding of the matrix of test projections V by the equation:

$$V = F_0 \Lambda . \tag{6}$$

When the matrix V has been obtained, a plot is made for each pair of columns. In the present example this results in ten graphs as shown in Figures 1 and 2. The foregoing procedure is followed for all trials except that the entries in the W matrix are derived from the diagrams rather than from the table of first factor residuals.

The entries in the W matrix of Trial 2 shown in Table 8 were derived from the graphs for Trial 1 shown in Figures 1 and 2. The graphs of factor A with the other factors were inspected in setting up the column A of W. The points 2, 14, 4, 3, and 23 are plainly high on factor A in all four graphs involving A. On the graph A-D, points 6 and 8 clearly seem to deserve high loadings on A. The con-

figuration of points in the E direction on the A-E graph seems too wide for all of the tests to have vanishing loadings on factor A and thus tests *12* and *24* are included in the high group on factor A. The case of tests 1 and 5 in the B direction on the A-B graph is more ambiguous; however, it was decided to include them also in the high group on factor A. Tests 21 and 25 on the A-C graph were left in the zero group on factor A. The tests placed in the high group have unities entered in the A column of the W matrix of Table 8. The cells for the tests in the zero group are left blank. Each of the other columns in W were obtained in the same way by inspecting the graphs for each of these factors. The selection of the subgroups is the high point in the procedure for Trial 2, since the computational procedure is identical in form to that outlined in the preceding paragraph for Trial 1.

When the computations for Trial 2 are completed and the graphs made, it is found that all of the subgroups are satisfactory. Two of these graphs are illustrated in Figure 3; the graph A-E seems to be the poorest of the ten and C-D the best. In order to tighten the points to the planes a third trial is made in which the entries in the W matrix of Table 9 for the high tests are taken as equal to the v's of Trial 2 rounded off to one decimal. Again the cells for the zero group are left blank. The computations are again the same as for Trial 1. The results of this trial are very near the ideal solution indicated in the original simple configuration of Table 1 except that the columns are in a different order. The order of the columns is of no consequence since they can be renamed and their order changed to any order desired. The graphs of Figure 4 illustrate the tightness of fit that has been obtained. All that is necessary to tighten the fit of the planes still further is to continue to another trial with improved weights in the W matrix taken from the present set of loadings in the V matrix of Table 9. However, since this was merely a repetition of the procedure already presented, it was not carried out.

A profitable variation in the procedure after Trial 1 is presented in Table 10 where the work of Trial 2′ is given. The difference between Trial 2 of Table 8 and Trial 2′ is in the use of differential weights in the W matrix of Trial 2′. These weights are assigned to the high tests from inspection of the graphs for the first trial which are presented in Figures 1 and 2. This was done at the time that the tests were placed in the high subgroup. Since the weights depend upon subjective estimates of final size of loadings from these graphs, it is probably best not to use too complicated or extensive a set. Therefore, it was decided to restrict these weights to the digits 1, 2, or 3. Tests 2, 14, and 4 are plainly higher than all of the other tests on factor A and are, therefore, given weights of 3. Tests 3 and 23

are given weights of 2, as are tests 6 and 8. These latter two tests seem from the graph *A-D* to have disproportionately low loadings on Trial 1. The other four tests in the high group, tests 1, 5, 12, and 24, are given weights of unity. The results of this trial are closer to the exact solution than are those of Trial 2, as will readily be seen by inspection of the *V* matrices of the two trials.

In order to gauge further the amount of discrepancy of each trial from the exact solution, the differences between the loadings in each trial and those in the exact solution were found and frequency distributions of the absolute values of these errors made. These distributions are shown in Table 11. It will be noted that each of the trials in the series 1, 2, and 3 are improvements over the preceding trials. Trial 2′, where subjective differential weights are used, is definitely better than Trial 2, being almost as good as Trial 3. The use of simple differential weights is recommended because of this gain. Further reassurance can be obtained from the fact that errors in these weights are not a too serious matter, for the goodness of results should not regress any further than if all of the tests were given unit weights as in Trial 2. It merely retards progress.

The advantage of using the graphs in selecting the subgroups can be illustrated by the initial selection of a high subgroup which includes tests high on two factors, *B* and *C* in this case. In Table 12 an alternate factor *B′* for Trial 1 is presented. It is to be imagined that this factor replaces the *B* factor in Table 7. The graphs between this new factor and the other four factors of Trial 1 are presented in Figure 5. The most profitable diagram for selecting the next high subgroup is *B′-C*, where there are no tests holding the *B* plane in the direction of *C*. A line for the new plane can be drawn on this diagram and tests selected which are farther from it. In this way the factors are guided to their proper solutions.

The computational procedure can be derived by either of two mathematical rationales. In the first of these the criterion is defined as:

$$y_{jp} \equiv (w_{jp} - k\, v_{jp}), \tag{7}$$

and

$$\sum_j y_{jp}^2 = \text{a minimum}. \tag{8}$$

The subscript p is used to designate rotated factor; thus w_{jp} is the entry for test j and factor p in the W matrix. v_{jp} is the resulting projection of test j on factor p. The w_{jp}'s are in this rationale to be thought of as proportional to a hypothetical set of loadings which might be obtained at the solution. The y_{jp}'s are the errors in fitting an actual set of loadings to these hypothetical ones. k is used so as to

consider only the relative order of magnitude of the v_{jp}'s. This solution has been derived by Mosier (2), where the w_{jp}'s were obtained from previous factorial studies. In this case they are derived from inspection of the configuration of the battery being investigated. Equation (8) is a least-squares statement for the errors y_{jp}.

A matrix equation for the v_{jp}'s is given in (6). In summation form this is:

$$v_{jp} = \sum_m a_{jm}\, \lambda_{mp} \tag{9}$$

where a_{jm} is the entry for test j and reference factor m in the original factor matrix F_0. λ_{mp} is the direction cosine for rotated factor p from the reference factor m. In order to facilitate the derivation it is convenient to define:

$$u_{jp} \equiv k\, v_{jp}; \tag{10}$$

for, then

$$u_{jp} = k \sum_m a_{jm}\, \lambda_{mp},$$

$$u_{jp} = \sum_m a_{jm}\, k\, \lambda_{mp},$$

or

$$u_{jp} = \sum_m a_{jm}\, l_{mp}, \tag{11}$$

where

$$l_{mp} \equiv k\, \lambda_{mp}. \tag{12}$$

The l_{mp}'s are direction numbers and are mutually independent. Equation (7) then becomes

$$y_{jp} = w_{jp} - u_{jp}. \tag{13}$$

In order to find the minimum solution of equation (8), the partial derivatives with respect to the l_{mp}'s are found and equated to zero. That is:

$$\frac{\partial \sum_j y^2_{jp}}{\partial\, l_{mp}} = 2 \sum_j y_{jp} \frac{\partial\, y_{jp}}{\partial\, l_{mp}};$$

using equation (13),

$$\frac{\partial \sum_j y^2_{jp}}{\partial\, l_{mp}} = 2 \sum_j (w_{jp} - u_{jp}) \left(- \frac{\partial\, u_{jp}}{\partial\, l_{mp}} \right),$$

and by (11),

$$\frac{\partial \sum_j y^2_{jp}}{\partial\, l_{mp}} = 2 \sum_j (w_{jp} - \sum_M a_{jM}\, l_{Mp})\, (-\, a_{jm}) = 0. \tag{14}$$

M is used as an alternative subscript for m. Equation (14) can be simplified to:

$$-\sum_j w_{jp}\, a_{jm} + \sum_j \sum_M a_{jM}\, l_{Mp}\, a_{jm} = 0,$$

or

$$\sum_j w_{jp}\, a_{jm} = \sum_j \sum_M a_{jM}\, l_{Mp}\, a_{jm},$$

or

$$\sum_j w_{jp}\, a_{jm} = \sum_M l_{Mp} \sum_j a_{jM}\, a_{jm}. \tag{15}$$

Equation (15) in matrix form is:

$$\boldsymbol{F}_0'\, \boldsymbol{W} = \boldsymbol{F}_0'\, \boldsymbol{F}_0\, \boldsymbol{L},$$

or by equations (2) and (3), becomes:

$$\boldsymbol{B} = \boldsymbol{P}\, \boldsymbol{L},$$

or

$$\boldsymbol{P}^{-1}\, \boldsymbol{B} = \boldsymbol{L}. \tag{4}$$

Equation (4) is the solution for minimizing the discrepancies between a hypothetical set of values given in $\boldsymbol{W}$ and values proportional by columns to the entries in $\boldsymbol{V}$. The factors approach the desired rotated solution as the w_{jp}'s are brought closer to the final v_{jp}'s. The discrepancies of the tests in the zero subgroup are constantly minimized since their w's are always set at zero.

The second rationale from which the computational procedure can be obtained is slightly more complicated than the first, but throws some additional light on the nature of the solution. This rationale starts from Horst's criterion in equation (1). If all of the tests are either h tests or z tests,

$$\sum_j v^2_{jp} = \sum_h v^2_{hp} + \sum_z v^2_{zp}; \tag{16}$$

then

$$\sum_z v^2_{zp} = \sum_j v^2_{jp} - \sum_h v^2_{hp},$$

and

$$\frac{\sum_h v^2_{hp}}{\sum_z v^2_{zp}} = \frac{\sum_h v^2_{hp}}{\sum_j v^2_{jp} - \sum_h v^2_{hp}},$$

or

$$\frac{\sum_h v^2_{hp}}{\sum_z v^2_{zp}} = \frac{1}{\dfrac{\sum_j v^2_{jp}}{\sum_h v^2_{hp}} - 1}. \tag{17}$$

It is evident that when

$$\frac{\sum_h v^2_{hp}}{\sum_z v^2_{zp}} = \text{a maximum},$$

$$\frac{\sum_j v^2_{jp}}{\sum_h v^2_{hp}} - 1 = \text{a minimum};$$

then

$$\frac{\sum_h v^2_{hp}}{\sum_j v^2_{jp}} = \text{a maximum}. \tag{18}$$

This last equation provides a simpler basis for the development of both the exact solution and the approximate ones.

As in the first rationale, it is easier here to deal with direction numbers, l_{mp}'s, and values, u_{jp}'s, proportional to the v_{jp}'s. Substituting equation (10) in equation (18) and cancelling the $(1/k^2)$'s in the numerator and denominator,

$$\frac{\sum_h u^2_{hp}}{\sum_j u^2_{jp}} = \text{a maximum}. \tag{19}$$

It is convenient now to define a condition placed upon the l_{mp}'s; that is, it is defined that:

$$\sum_j u^2_{jp} \equiv 1. \tag{20}$$

With this condition it is necessary only to make

$$\sum_h u^2_{hp} = \text{a maximum}. \tag{21}$$

However, instead of continuing to the exact solution, consider the following equation:

$$\sum_j u_{jp} w_{jp} = \text{a maximum}. \tag{22}$$

When the w_{jp}'s are zero for the z tests and equal to the u_{hp}'s for the h tests, this equation becomes the same as (21). However, when the w_{hp}'s are plus or minus one, the signs being taken as the same as the u_{hp}'s, equation (22) becomes

$$\sum_h | u_{hp} | = \text{a maximum}. \tag{23}$$

Thus, with proper selection of the w_{hp}'s, the weighted sum of the u_{jp}'s becomes either the sum of the squares of the u_{hp}'s or the sum of the absolute values of the u_{hp}'s. This latter squared is used as approximately proportional to the former. That is:

$$[\sum_h |u_{hp}|]^2 \simeq c \sum_h u^2_{hp}. \tag{24}$$

The solution of equation (22) under the condition of equation (20) is accomplished with the use of LaGrange multipliers γ_p. Thus

$$\frac{\partial \sum_j u_{jp} w_{jp}}{\partial l_{mp}} + \gamma_p \frac{\partial (\sum_j u^2_{jp} - 1)}{\partial l_{mp}} = 0. \tag{25}$$

Then

$$\sum_j w_{jp} \frac{\partial u_{jp}}{\partial l_{mp}} + 2 \gamma_p \sum_j u_{jp} \frac{\partial u_{jp}}{\partial l_{mp}} = 0;$$

or, using equation (11)

$$\sum_j w_{jp} a_{jm} + 2 \gamma_p \sum_j \sum_M a_{jM} l_{Mp} a_{jm} = 0,$$

or

$$\sum_j w_{jp} a_{jm} = -2 \gamma_p \sum_M l_{Mp} \sum_j a_{jM} a_{jm}. \tag{26}$$

However, since each column of $\boldsymbol{L}$ is to be separately normalized, the multipliers $(-2 \gamma_p)$ can be dropped and equation (26) becomes identical with equation (15) and reduces to equation (4) in a similar manner.

Since equation (24) is usually a good type of approximation, it was expected that the use of w_{hp}'s equal to unity should give good results. This expectation is at least partially born out by the results illustrated in Trials 1 and 2 presented in Tables 7 and 8. Better approximations can be obtained by using differential weights as in Trial 2′ in Table 10 since these w_{hp}'s approach closer to being proportional to the solution v_{hp}'s.

Although the method was illustrated with a problem having a positive manifold, it is not restricted to such a problem. Tests with negative loadings can be entered into the high group by using negative w_{hp}'s. The only portion of the procedure that is not generally applicable is in the selection of the first trial subgroups. Some other principle than the one used here may have to be used in the case where there are tests with significant negative factor loadings.

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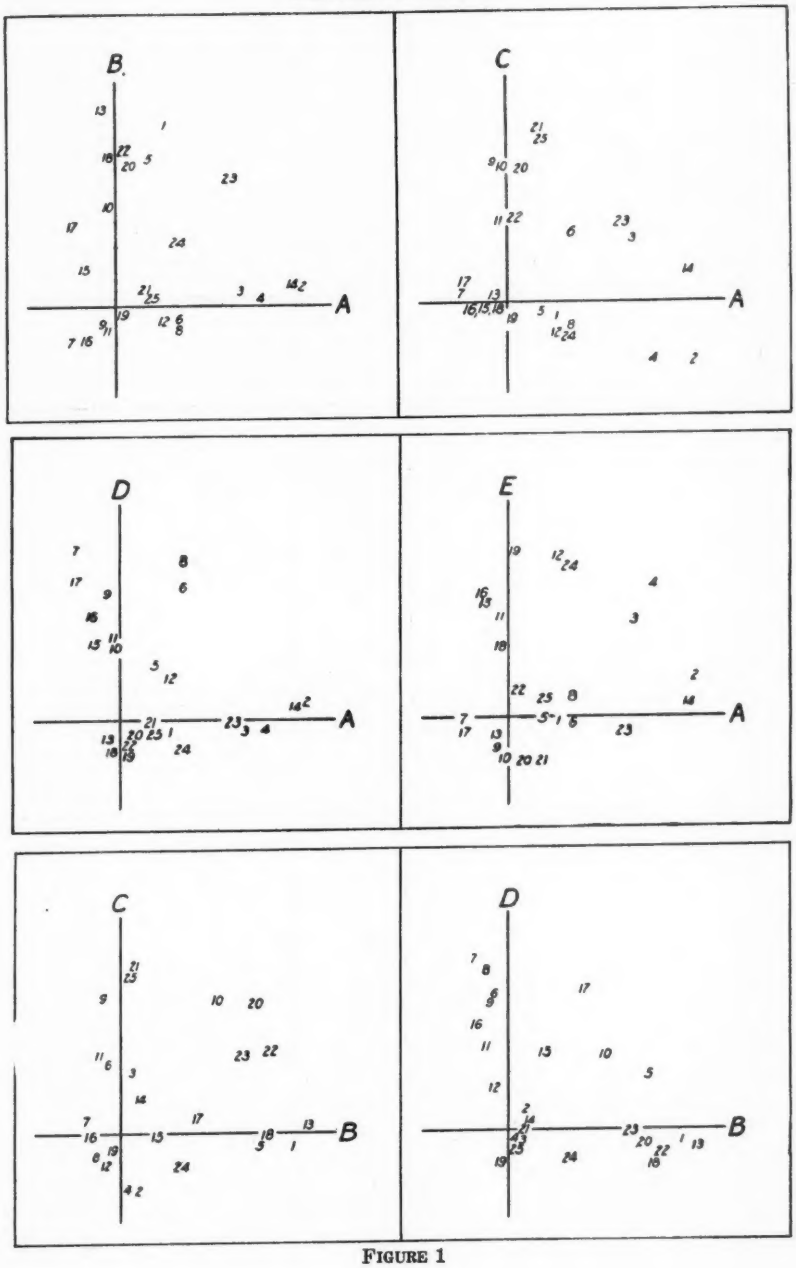


FIGURE 1

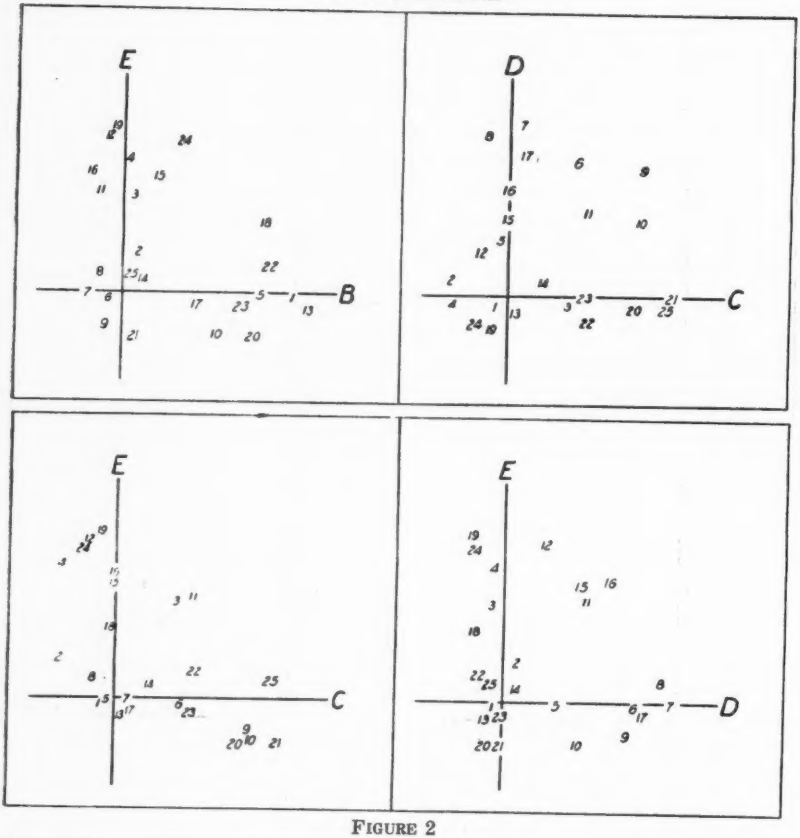
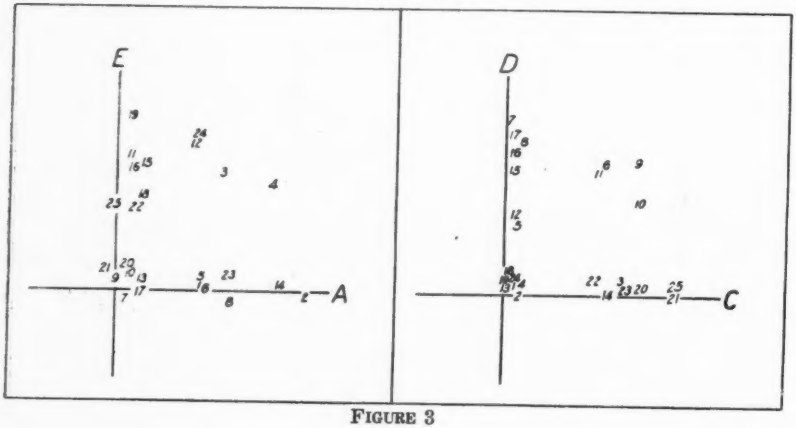


FIGURE 2



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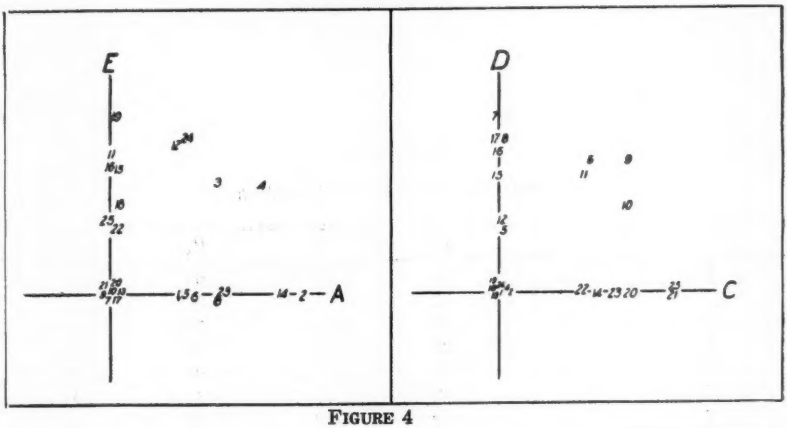
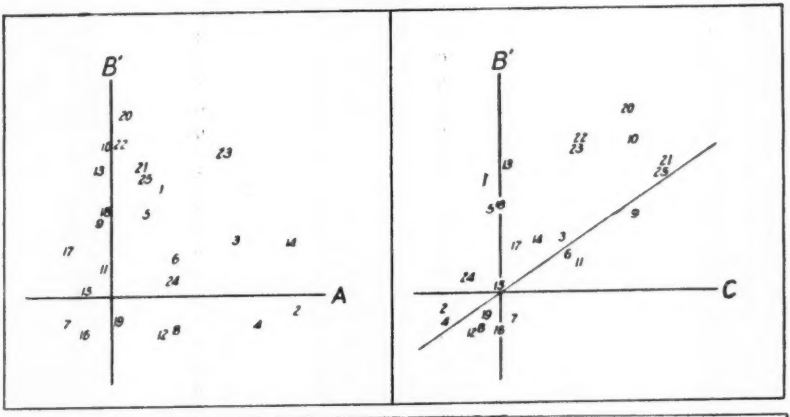


FIGURE 4



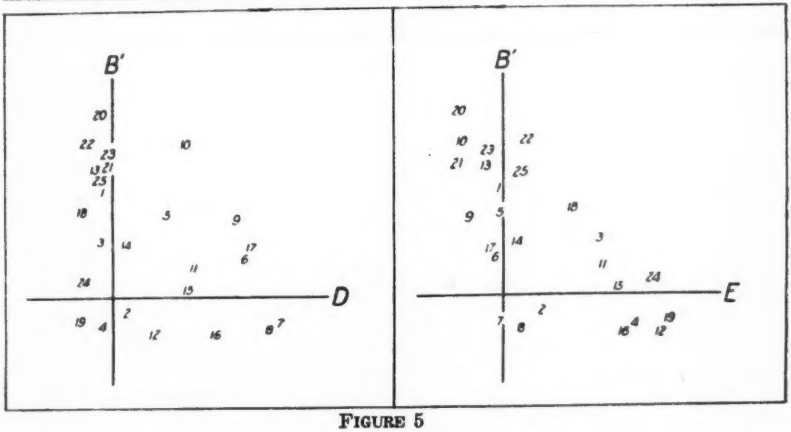


FIGURE 5

TABLE 1

A Fictitious Simple Configuration

	P_1	P_2	P_3	P_4	P_5
1	.0	.3	.0	.0	.8
2	.0	.9	.0	.0	.0
3	.5	.5	.0	.5	.0
4	.5	.7	.0	.0	.0
5	.0	.3	.3	.0	.7
6	.0	.4	.6	.4	.0
7	.0	.0	.8	.0	.0
8	.0	.5	.7	.0	.0
9	.0	.0	.6	.6	.0
10	.0	.0	.4	.6	.5
11	.6	.0	.5	.4	.0
12	.7	.3	.3	.0	.0
13	.0	.0	.0	.0	.9
14	.0	.8	.0	.4	.0
15	.6	.0	.5	.0	.3
16	.6	.0	.6	.0	.0
17	.0	.0	.7	.0	.5
18	.4	.0	.0	.0	.7
19	.8	.0	.0	.0	.0
20	.0	.0	.0	.6	.6
21	.0	.0	.0	.8	.0
22	.3	.0	.0	.4	.7
23	.0	.5	.0	.5	.5
24	.7	.3	.0	.0	.3
25	.3	.0	.0	.8	.0

TABLE 2

Centroid Factor Matrix, F_0

	I	II	III	IV	V
1	.51	—.19	.53	.33	—.22
2	.37	.49	.54	—.08	.38
3	.65	.47	—.03	—.30	—.08
4	.51	.31	.37	—.48	.15
5	.60	—.27	.37	.32	.02
6	.61	.18	—.21	.17	.45
7	.36	—.35	—.30	.10	.54
8	.52	—.03	.04	.05	.69
9	.54	.07	—.56	.26	.22
10	.69	—.06	—.26	.45	—.13
11	.67	—.08	—.46	—.31	.05
12	.57	—.07	.00	—.57	.13
13	.44	—.39	.39	.39	—.39
14	.51	.66	.25	.06	.21
15	.64	—.44	—.11	—.30	.04
16	.54	—.35	—.28	—.42	.24
17	.56	—.52	—.04	.31	.26
18	.52	—.37	.27	—.03	—.42
19	.36	—.12	—.07	—.67	—.23
20	.56	.07	—.07	.45	—.45
21	.36	.44	—.44	.25	—.25
22	.65	—.13	.06	.18	—.51
23	.67	.33	.24	.33	—.16
24	.58	—.07	.25	—.48	—.20
25	.49	.40	—.47	.00	—.33

TABLE 3
First Centroid Residuals *

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1		08	−19	−05	35	−19	−18	−11	−28	05	−35	−20	50	−02	−08	−28	12	30	−19	20	−18	23	21	03	−26
2	08		21	45	05	13	−14	27	−21	−26	−25	06	−16	54	−24	−20	−20	−19	−13	−21	−13	−24	20	06	−18
3	−19	21		27	−23	01	−23	−08	−04	−14	06	13	−28	27	−11	−05	−36	−14	16	−06	17	−08	06	12	23
4	−05	45	27		−10	−04	−18	08	−28	−35	−04	27	−22	30	−03	02	−28	−06	22	−29	−18	−18	01	27	−10
5	35	05	−23	−10		−06	02	05	−14	06	−25	−16	37	−06	−02	−14	23	18	−21	09	−21	10	10	−05	−29
6	−19	13	01	−04	−06		26	31	27	06	05	−05	−26	17	−09	03	08	−32	−22	−10	10	−23	−01	−23	02
7	−18	−14	−23	−18	02	26		38	29	07	16	04	−15	−18	18	29	36	−18	−12	−20	−13	−23	−24	−20	−17
8	−11	27	−08	08	05	31	38		14	−08	00	06	−22	14	02	14	21	−27	−19	−29	−19	−34	−10	−15	−26
9	−28	−21	−04	−28	−14	27	29	14		23	18	−12	−23	−03	−04	08	12	−28	−19	06	29	−11	−06	−31	22
10	05	−26	−14	−35	06	06	07	−08	23		−02	−27	15	−10	−08	−12	14	−01	−24	27	23	14	09	−25	14
11	−35	−25	06	−04	−25	05	16	00	18	−02		19	−29	−17	18	30	−02	−11	24	−13	08	−10	−24	03	17
12	−20	06	13	27	−16	−05	04	06	−12	−27	19		−25	−05	20	29	−11	−01	36	−32	−20	−16	−23	25	−07
13	50	−16	−28	−22	37	−26	−15	−22	−23	15	−29	−25		−22	00	−23	20	40	−15	29	−15	34	15	02	−21
14	−02	54	27	30	−06	17	−18	14	−03	−10	−17	−05	−22		−33	−27	−28	−26	−18	−04	14	−17	26	−05	08
15	−08	−24	−11	−03	−02	−09	18	02	−04	−08	18	20	00	−33		32	15	12	25	−17	−23	−02	−28	14	−14
16	−28	−20	−05	02	−14	03	29	14	08	−12	30	29	−23	−27	32		12	−03	29	−30	−19	−17	−36	11	−09
17	12	−20	−36	−28	23	08	36	21	12	14	−02	−11	20	−28	15	12		06	−20	−01	−20	−01	−12	−17	−28
18	30	−19	−14	−06	18	−32	−18	−27	−28	−01	−11	−01	40	−26	12	−03	06		14	13	−18	27	00	19	−14
19	−19	−13	16	22	−21	−22	−12	−19	−19	−24	24	36	−15	−18	25	29	−20	14		−20	−13	01	−24	36	06
20	20	−21	−06	−29	09	−10	−20	−29	06	27	−13	−32	29	−04	−17	−30	−01	13	−20		28	30	23	−15	21
21	−18	−13	17	−18	−21	10	−13	−19	29	23	08	−20	−15	14	−23	−19	−20	−18	−13	28		09	16	−21	47
22	23	−24	−08	−18	10	−23	−23	−34	−11	14	−10	−16	34	−17	−02	−17	−01	27	01	30	09		11	04	09
23	21	20	06	01	10	−01	−24	−10	−06	09	−24	−23	15	26	−28	−36	−12	00	−24	23	16	11		−09	07
24	03	06	12	27	−05	−23	−20	−15	−31	−24	03	25	02	−05	14	11	−17	19	36	−15	−21	04	−09		−08
25	−26	−18	23	−10	−29	02	−17	−26	22	14	17	−07	−21	08	−14	−09	−28	−14	06	21	47	09	07	−08	

* The entries in this table were computed with the Matrix Multiplier (8) and may be in error by .01. All decimal points were omitted from this table; each entry has two decimal places.

TABLE 4

Positive Clusters of First Factor Residuals

A	*B*	*C*	*D*	*E*
2	1	21	7	12
14	13	25	8	19
4	18	9	17	24
3	5	20	6	4
23	22	10	9	16
	20		16	
	23			

TABLE 5

$P = F_0' F_0$

	I	II	III	IV	V
I	7.5128	—.0857	—.0568	.0768	—.0940
II	—.0857	2.6564	—.0002	—.0707	.1038
III	—.0568	—.0002	2.5033	.0704	—.2887
IV	.0768	—.0707	.0704	2.8753	—.3891
V	—.0940	.1038	—.2887	—.3891	2.5289

TABLE 6

P^{-1}

	I	II	III	IV	V
I	.1333	.0040	.0037	—.0029	.0048
II	.0040	.3773	—.0018	.0073	—.0144
III	.0037	—.0018	.4049	—.0039	.0458
IV	—.0029	.0073	—.0039	.3554	.0538
V	.0048	—.0144	.0458	.0538	.4097

TABLE 7
Trial 1

	W					B						V*				
	A	B	C	D	E		A	B	C	D	E		A	B	C	D	E
1		1				I	2.71	3.95	2.64	3.13	2.56	1	.23	.80	—.06	—.05	—.02
2	1					II	2.26	—.95	.92	—1.00	—.30	2	.86	.08	—.26	.07	.18
3	1					III	1.37	1.79	—1.80	—1.35	.27	3	.58	.05	.28	—.05	.44
4	1				1	IV	—.47	1.97	1.41	.47	—2.62	4	.68	.03	—.25	—.04	.61
5		1				V	.50	—2.13	—.94	2.40	.09	5	.16	.65	—.04	.25	—.01
6				1								6	.30	—.06	.31	.61	—.03
7				1								7	—.20	—.16	.06	.78	—.01
8				1								8	.30	—.10	—.10	.73	.09
9			1	1		I	.38	.51	.34	.42	.35	9	—.05	—.08	.62	.58	—.15
10			1			II	.85	—.30	.38	—.39	—.12	10	—.02	.44	.61	.34	—.19
11						III	.59	.64	—.77	—.43	.13	11	—.03	—.10	.36	.38	.46
12					1	IV	—.14	.56	.46	.29	—.94	12	.23	—.07	—.13	.19	.72
13		1				V	.22	—.65	—.39	.98	—.08	13	—.06	.87	.03	—.08	—.08
14	1											14	.83	.09	.16	.06	.06
15												15	—.12	.16	—.01	.35	.53
16				1	1							16	—.13	—.15	—.01	.48	.55
17				1		I	.34	.42	.31	.34	.34	17	—.20	.35	.07	.64	—.06
18		1				II	.75	—.25	.34	—.31	—.12	18	—.03	.66	—.02	—.14	.32
19					1	III	.52	.52	—.70	—.34	.13	19	.03	—.04	—.07	—.15	.76
20		1	1			IV	—.12	.46	.42	.23	—.92	20	.07	.62	.59	—.06	—.20
21			1			V	.19	—.53	—.35	.79	—.08	21	.15	.06	.76	—.01	—.20
22		1										22	.04	.69	.37	—.12	.12
23	1	1										23	.53	.56	.35	—.01	—.07
24					1							24	.29	.28	—.15	—.13	.69
25			1			A	1.00	.07	—.12	—.17	.19	25	.16	.04	.73	—.06	.08
						B	.07	1.00	.06	—.27	—.14						
						C	—.12	.06	1.00	.06	—.38						
						D	—.17	—.27	.06	1.00	—.17						
						E	.19	—.14	—.38	—.17	1.00						

L

	A	B	C	D	E
I	.38	.51	.34	.42	.35
II	.85	—.30	.38	—.39	—.12
III	.59	.64	—.77	—.43	.13
IV	—.14	.56	.46	.29	—.94
V	.22	—.65	—.39	.98	—.08

Λ

	A	B	C	D	E
I	.34	.42	.31	.34	.34
II	.75	—.25	.34	—.31	—.12
III	.52	.52	—.70	—.34	.13
IV	—.12	.46	.42	.23	—.92
V	.19	—.53	—.35	.79	—.08

C

	A	B	C	D	E
A	1.00	.07	—.12	—.17	.19
B	.07	1.00	.06	—.27	—.14
C	—.12	.06	1.00	.06	—.38
D	—.17	—.27	.06	1.00	—.17
E	.19	—.14	—.38	—.17	1.00

* The entries in the V matrix were computed on the Matrix Multiplier (8) and may be in error by .01.

TABLE 8
Trial 2

W

	A	B	C	D	E
1	1	1			
2	1				
3	1		1		1
4	1				1
5	1	1		1	
6	1		1	1	
7				1	
8	1			1	
9			1	1	
10		1	1	1	
11			1	1	1
12	1			1	1
13		1			
14	1		1		
15		1		1	1
16				1	1
17		1		1	
18		1			1
19					1
20		1	1		
21			1		
22		1	1		1
23	1	1	1		
24	1	1			1
25			1		1

B

	A	B	C	D	E
I	6.10	6.42	6.40	6.30	6.18
II	1.81	—2.04	2.35	—1.92	—.45
III	2.35	1.63	—1.95	—1.81	—.47
IV	—.65	1.95	1.54	.06	—3.38
V	1.37	—2.16	—.98	2.51	—1.16

L

	A	B	C	D	E
I	.84	.84	.85	.84	.82
II	.68	—.70	.94	—.73	—.15
III	1.04	.58	—.82	—.59	—.21
IV	—.17	.54	.50	.13	—1.28
V	.64	—.65	—.41	1.01	—.64

Λ

	A	B	C	D	E
I	.51	.56	.52	.52	.49
II	.41	—.47	.57	—.45	—.09
III	.63	.39	—.50	—.36	—.13
IV	—.10	.36	.30	.08	—.77
V	.39	—.43	—.25	.62	—.38

C

	A	B	C	D	E
A	.99	.13	.06	.09	.06
B	.13	1.00	.04	.12	.15
C	.06	.04	1.00	.06	.13
D	.09	.12	.06	.99	.04
E	.06	.15	.13	.04	1.00

*V**

	A	B	C	D	E
1	.40	.80	.05	.05	.03
2	.89	.00	.08	.01	—.02
3	.50	.06	.55	.06	.54
4	.73	.05	.08	.05	.49
5	.40	.71	.06	.34	.02
6	.41	.04	.46	.60	.01
7	.05	.05	.03	.80	—.04
8	.54	.04	.08	.70	—.05
9	.01	.05	.62	.61	.05
10	.07	.53	.62	.43	.08
11	.07	.10	.43	.56	.62
12	.37	.09	.05	.36	.68
13	.12	.89	.03	.05	.05
14	.76	.00	.48	.01	.03
15	.12	.40	.04	.57	.58
16	.09	.10	.04	.65	.56
17	.12	.54	.04	.73	—.01
18	.12	.74	.02	.08	.44
19	.07	.09	.01	.07	.80
20	.05	.61	.61	.04	.10
21	—.04	.02	.80	.01	.10
22	.10	.73	.42	.07	.38
23	.53	.50	.56	.04	.07
24	.39	.37	.04	.08	.70
25	—.01	.05	.80	.04	.39

* The entries in the V matrix were computed on the Matrix Multiplier (8) and may be in error by .01.

TABLE 9
Trial 3

W

	A	B	C	D	E
1	.4	.8			
2	.9				
3	.5		.6		.5
4	.7				.5
5	.4	.7		.3	
6	.4		.5	.6	
7				.8	
8	.5			.7	
9			.6	.6	
10		.5	.6	.4	
11			.4	.6	.6
12	.4			.4	.7
13		.9			
14	.8		.5		
15		.4		.6	.6
16				.6	.6
17		.5		.7	
18		.7			.4
19					.8
20		.6	.6		
21			.8		
22		.7	.4		.4
23	.5	.5	.6		
24	.4	.4			.7
25			.8		.4

B

	A	B	C	D	E
I	3.166	3.827	3.634	3.528	3.447
II	1.403	−1.329	1.536	−1.170	−.366
III	1.446	1.249	−1.276	−1.205	−.277
IV	−.412	1.447	.977	.020	−2.219
V	.912	−1.513	−.678	1.703	−.504

L

	A	B	C	D	E
I	.44	.50	.48	.47	.46
II	.52	—.46	.61	—.45	—.13
III	.64	.45	—.54	—.39	—.11
IV	—.10	.41	.32	.08	—.83
V	.41	—.45	—.29	.68	—.32

Λ

	A	B	C	D	E
I	.43	.49	.46	.46	.45
II	.51	—.45	.59	—.44	—.13
III	.62	.44	—.52	—.38	—.11
IV	—.10	.40	.31	.08	—.82
V	.40	—.44	—.28	.67	—.32

C

	A	B	C	D	E
A	1.00	.04	.03	.00	.01
B	.04	.99	—.02	—.01	.04
C	.03	—.02	1.00	—.01	.02
D	.00	—.01	—.01	1.00	.03
E	.01	.04	.02	.03	1.01

*V**

	A	B	C	D	E
1	.33	.80	.01	.00	.00
2	.90	.00	.05	.00	—.01
3	.50	.01	.52	.03	.51
4	.71	.02	.03	.02	.49
5	.33	.70	.02	.29	.00
6	.39	.00	.42	.60	—.01
7	.00	.00	—.01	.80	—.01
8	.50	.00	.02	.70	—.03
9	—.02	—.01	.60	.60	.01
10	.01	.49	.59	.39	.02
11	.01	.02	.39	.53	.60
12	.32	.03	.01	.33	.69
13	.04	.89	.00	.00	.01
14	.79	—.01	.45	—.01	.00
15	.03	.33	—.01	.53	.59
16	.02	.03	—.01	.64	.59
17	.02	.50	—.01	.70	—.01
18	.04	.71	—.01	.02	.41
19	.03	.03	—.01	.04	.81
20	.01	.59	.60	—.01	.03
21	—.02	—.01	.80	—.01	.03
22	.03	.70	.39	.01	.32
23	.51	.49	.53	—.01	.01
24	.34	.32	.00	.03	.70
25	—.01	.00	.80	.01	.33

* The entries in the V matrix were computed on the Matrix Multiplier (8) and may be in error by .01.

TABLE 10
Trial 2′

W

	A	B	C	D	E
1	1	3			
2	3				
3	2		2		2
4	3				3
5	1	3		1	
6	2		2	3	
7				3	
8	2			3	
9			2	3	
10		2	2	2	
11			2	2	2
12	1			1	3
13		3			
14	3		2		
15		1		2	2
16				2	2
17		2		3	
18		3			2
19					3
20		2	2		
21			3		
22		3	2		1
23	2	2	2		
24	1	1			3
25			3		1

B

	A	B	C	D	E
I	11.33	14.34	13.65	14.02	13.24
II	5.68	—4.92	5.54	—4.15	—1.12
III	4.71	4.74	—4.81	—5.06	.02
IV	—1.40	5.87	3.33	1.26	—9.14
V	3.75	—5.68	—2.54	7.03	—1.63

L

	A	B	C	D	E
I	1.57	1.87	1.80	1.86	1.78
II	2.12	—1.68	2.21	—1.59	—.41
III	2.12	1.70	—2.04	—1.67	.02
IV	—.31	1.68	1.07	.77	—3.38
V	1.65	—1.65	—1.10	2.84	—1.08

Λ

	A	B	C	D	E
I	.42	.49	.47	.45	.45
II	.56	—.44	.58	—.38	—.10
III	.56	.44	—.53	—.40	.01
IV	—.08	.44	.28	.18	—.85
V	.44	—.43	—.29	.68	—.27

C

	A	B	C	D	E
A	1.00	—.02	.08	.04	.09
B	—.02	1.01	—.01	.00	.01
C	.08	—.01	1.00	.06	—.01
D	.04	.00	.06	1.00	—.10
E	.09	.01	—.01	—.10	1.01

V*

	A	B	C	D	E
1	.28	.80	.00	.00	.03
2	.91	.01	.04	.01	.09
3	.51	.00	.53	.02	.52
4	.70	.00	.05	—.02	.57
5	.30	.71	.01	.29	.02
6	.42	.01	.42	.63	—.01
7	.02	.01	.00	.80	—.04
8	.52	.01	.02	.71	.01
9	.03	.01	.60	.64	—.05
10	.02	.50	.59	.43	—.03
11	.02	.01	.42	.50	.56
12	.30	.01	.03	.27	.72
13	—.02	.90	.00	—.01	.01
14	.81	.00	.45	.03	.06
15	.00	.31	.01	.47	.57
16	.01	.01	.02	.57	.56
17	.01	.51	.00	.69	—.03
18	—.02	.70	.00	—.03	.41
19	.00	.00	.02	—.04	.80
20	.00	.60	.60	.03	—.02
21	.02	.01	.80	.05	—.03
22	—.01	.70	.40	.00	.29
23	.50	.50	.52	.03	.04
24	.29	.30	.03	—.03	.73
25	.02	.00	.81	.04	.27

* The entries in the V matrix were computed on the Matrix Multiplier (8) and may be in error by .01.

TABLE 11

Distributions of Absolute Discrepancies

Discrepancy	*Trial 1*	*Trail 2*	*Trail 2'*	*Trial 3*
.26	1			
.25	1			
.24	1			
.23				
.22	2			
.21				
.20	5			
.19	1			
.18	2			
.17				
.16	2			
.15	8			
.14	4			
.13	3			
.12	4	4		
.11	2			
.10	4	8		
.09	3	5	1	
.08	6	8		
.07	10	9	1	
.06	15	7	1	
.05	7	16	4	2
.04	10	19	8	5
.03	11	11	21	20
.02	10	10	22	16
.01	12	16	33	50
.00	1	12	34	32

TABLE 12
Plane B′ for Trial 1

	W		*B*		*V**
1		I	2.57	1	.49
2		II	.21	2	—.07
3		III	—.03	3	.25
4		IV	1.41	4	—.13
5		V	—1.25	5	.38
6				6	.17
7			*L*	7	—.12
8				8	—.15
9		I	.33	9	.35
10	1	II	.12	10	.69
11		III	—.07	11	.13
12		IV	.43	12	—.17
13		V	—.43	13	.58
14				14	.24
15			A	15	.03
16				16	—.17
17		I	.47	17	.21
18		II	.17	18	.39
19		III	—.10	19	—.11
20	1	IV	.61	20	.83
21		V	—.61	21	.59
22	1			22	.70
23	1		*C*	23	.65
24				24	.07
25		A	.05	25	.55
		B′	1.00		
		C	.74		
		D	—.20		
		E	—.39		

* The entries in the V matrix were computed on the Matrix Multiplier (8) and may be in error by .01.

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RESEARCH NOTE
IN SEARCH OF NEW TESTS

We should like to ask the assistance of psychologists in our search for a particular kind of test. Most tests are of the kind that give a unimodal distribution in which even the poorest subject makes some appreciable performance and in which the ratio between the best and the poorest is about 2 to 1, or 3 to 1. Other things being equal it is expected that if a test is factorially relatively pure (low complexity) then the individual differences will be conspicuous. Some subjects endowed with the factor involved will find the task easy and they will even take some pleasure in doing it. Others who are not so endowed will find the task not only difficult and disagreeable but they may even refuse to continue trying. We are looking for tests which can be easily explained to untutored adults and which show these gross individual differences. It is not a question of whether the ability that is involved is educationally or socially important. That is irrelevant. Nor do we care at all whether the distribution is normal or skewed. That is entirely irrelevant. We are looking for tasks that are easily explained to adults and which show gross individual differences. Such tests are likely to be relatively pure factorially and they should be significant in isolating the fundamental parameters of mental function. Only occasionally do we find such tasks.

We shall appreciate it if others who discover such simple tasks will let us know so that more tests of that kind may be incorporated in factorial studies.

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